Reverse Data-Processing Theorems

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what is a data-processing theorem?

Let $\mathcal{N} : X \to Y$ and $\mathcal{W} : Y \to Z$ be two noisy channels, then, the joint distributions $(U, Y) \equiv (U, \mathcal{N}(X))$ and $(U, Z) \equiv (U, \mathcal{W}(Y))$ are such that

$$H(U|Y) \leq H(U|Z) \quad \text{for all initial} \quad (U, X)$$
Let $\mathcal{B} : \mathcal{X} \to (\mathcal{Y}, \mathcal{Z})$ be a broadcast channel. Assume that the final distribution $(U, Y, Z) \equiv (U, B(X))$ is such that

$$H(U|Y) \leq H(U|Z) \quad \text{for all initial} \quad (U, X)$$

Can we then conclude that there exists a noisy channel $\mathcal{W} : \mathcal{Y} \to \mathcal{Z}$ such that $(U, Z) = (U, W(Y))$ for all initial $(U, X)$?

No (Körner and Marton, 1977)
Given two noisy channels $\mathcal{N} : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{N}^\prime : \mathcal{X} \to \mathcal{Z}$, Körner and Marton (1977) introduce the following definitions:

- $\mathcal{N}$ is **degradable** into $\mathcal{N}^\prime$ if there exists a noisy channel $\mathcal{W} : \mathcal{Y} \to \mathcal{Z}$ such that $\mathcal{N}^\prime = \mathcal{W} \circ \mathcal{N}$.

- $\mathcal{N}$ is **less noisy** than $\mathcal{N}^\prime$ if $H(U|Y) \leq H(U|Z)$ for all initial $(U, X)$.

- $\mathcal{N}$ is **more capable** than $\mathcal{N}^\prime$ if $H(X|Y) \leq H(X|Z)$ for all initial $X$.

**Fact:** degradable $\iff$ less noisy $\implies$ more capable.
a reverse that works

Reverse Data-Processing Theorem (FB, Prob. Inf. Trans. 2016)

Given two noisy channels \( \mathcal{N} : \mathcal{X} \rightarrow \mathcal{Y} \) and \( \mathcal{N}' : \mathcal{X} \rightarrow \mathcal{Z} \),

\[
H_{\min}(U|Y) \leq H_{\min}(U|Z)
\]

for all initial \((U, X)\), if and only if \( \mathcal{N} \) is degradable in \( \mathcal{N}' \). In other words,

degradable \( \iff \) less noisy w.r.t. \( H_{\min} \).

\[H_{\min}(U|Y) = -\log_2 P_{\text{guess}}(U|Y) = -\log_2 \sum_y \max_u p(u, y)\]

\[\text{it also holds approximately}\]
finding stochastic dependencies

✓ suppose that, given a conditional probability \( p(y_1, y_2, \cdots, y_N \mid x) \), we want to find stochastic dependencies between these variables

reverse data-processing theorem: a path exists if and only if \( H_{\text{min}} \) never decreases (equivalently, no path exists if and only if \( H_{\text{min}} \) strictly decreases at some point for some initial conditions)

stochastic dependencies follow the “flow” of \( H_{\text{min}} \)

partial ordering \( \implies \) incomparable paths

for example, the figure above is equivalent to the following entropic conditions:

\[
\begin{align*}
H_{\text{min}}(U \mid Y_1) &\leq H_{\text{min}}(U \mid Y_2) \leq H_{\text{min}}(U \mid Y_4) \leq H_{\text{min}}(U \mid Y_5), \\
H_{\text{min}}(U \mid Y_3) &\leq H_{\text{min}}(U \mid Y_5), \\
H_{\text{min}}(U \mid Y_3) &\leq \{H_{\text{min}}(U \mid Y_1), H_{\text{min}}(U \mid Y_2), H_{\text{min}}(U \mid Y_4)\},
\end{align*}
\]

for all initial \((U, X)\)

for all initial \((U, X)\)

for some initial \((U, X)\)
divisibility of stochastic processes

- divisibility: $N_i = \mathcal{W}_i \circ \mathcal{W}_{i-1} \circ \cdots \circ \mathcal{W}_1$

- a dynamical map is divisible if and only if the sequence $\{H_{\min}(U|Y_i)\}_{i \geq 1}$ is non-decreasing for all initial $(U, X)$

- namely, divisibility is equivalent to “no $H_{\min}$ backflow”

- the same insight holds also in the quantum (Buscemi, Datta; PRA 2016) approximate (Jencova; ISIT 2016) case
the case of open quantum systems dynamics

\( S \): system, \( E \): environment, \( S + E \): conservative

But the given process need not be collisional to be divisible. **Question:** how to characterize system-environment correlations that do not break divisibility?
exponentially, the initial factorization condition is an approximation (and strong-coupling regimes are of interest)

what happens to the reduced dynamics in the presence of initial system-environment correlations?

Pechukas (PRL, 1994): “Here we show that complete positivity is an artifact of product initial conditions. In general, reduced dynamics need not be CP”

Lindblad (J. Phys. A, 1995) and Alicki (PRL, 1995, comment to Pechukas)

Rodriguez-Rosario, Modi, Kuah, Sudarshan (2008): null discord $\Rightarrow$ divisibility

Shabani–Lidar (2009): null discord $\iff$ divisibility (Erratum 2016)

Brodutch, Datta, Modi, Rivas, Rodriguez-Rosario (2013): divisibility $\iff$ null discord

FB (2014): what follows
the way I see it

- System and environment start in a correlated state: they have to be considered as one composite system.
- Reference–system–environment: $\mathcal{H}_U \otimes \mathcal{H}_S \otimes \mathcal{H}_E$
- The initial condition is given as a tripartite density operator $\rho_{USE}$.

The state $\rho_{USE}$ is "passive" if $H(U|S)_\rho \leq H(U|S')_\sigma$ for all isometries $V$.

**First fact:** $\rho_{USE}$ passive $\iff \forall V_{SE}, \exists \mathcal{E}_S^V: \sigma_{US'} = (\text{id}_U \otimes \mathcal{E}_S^V)(\rho_{US})$

**Second fact:** $\rho_{USE}$ passive $\iff I(U; E|S)_\rho = 0 \implies \text{"passive"}\equiv\text{"Markovian"}$

The above facts constitute another reverse data-processing theorem.

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conventional approach

- no reference system; instead, given is a family $\mathcal{I} = \{\rho_{SE} : \rho_{SE} \in \mathcal{I}\}$ of possible initial joint system–environment states

$$
\begin{array}{ccc}
\mathcal{S}_{SE} & \xrightarrow{V: SE \rightarrow S'E'} & \mathcal{S}'_{S'E'} \\
\downarrow{\text{Tr}_E} & & \downarrow{\text{Tr}_{E'}} \\
\mathcal{S}_S & \xrightarrow{\xi_V: S \rightarrow S'} & \mathcal{S}'_S \\
\end{array}
$$

- in the above diagram, $\rho_{SE}$ is a generic element of $\mathcal{I}$ (i.e., the condition must hold for all $\rho_{SE} \in \mathcal{I}$)

- if the above diagram holds for all isometries $V$, we say that the family $\mathcal{I}$ is Markovian

- example: the initial factorization condition, i.e., $\mathcal{I} = \{\rho_S \otimes \bar{\xi}_E : \rho_S \text{ any state of } S\}$ for some fixed environment state $\bar{\xi}_E$, defines a Markovian family
connecting the two approaches

- how to connect the tripartite with the bipartite scenario?
- steering: $\rho_{SE} = \frac{\text{Tr}_U[\rho_{USE} (\Pi_U \otimes 1_{SE})]}{\text{Tr}[\rho_{USE} (\Pi_U \otimes 1_{SE})]}$ for some $\Pi > 0$
- we say that the family $\mathcal{I}$ is steerable if there exists a $\rho_{USE}$ such that:
  1. for all $\rho_{SE} \in \mathcal{I}$ there exists a $\Pi_U > 0$ such that the above holds, and
  2. for all $\Pi_U > 0$, the steered state $\rho_{SE}^{\Pi}$ is in $\mathcal{I}$
- example: if $\mathcal{I}$ is a polytope (e.g., bipartite cq-states) then it’s steerable
- example: the family $\mathcal{I} = \{\rho_S \otimes \xi_E\}$ corresponding to the initial factorization condition is steerable
- main fact: bipartite family $\mathcal{I}$ is steerable and Markovian $\iff$ it can be steered from Markovian $\rho_{USE}$, i.e., $I(U; E|S)_\rho = 0$
- first corollary: all previous cases (they all happen to be steerable Markovian families)
- many other more general constructions are possible too $\implies$ no direct connection between “strength/character of initial correlations” and “existence of CPTP reduced dynamics”
- second corollary: if $\mathcal{I}$ is Markovian, steerable, and such that $\text{Tr}_E[\mathcal{I}]$ contains all states on $\mathcal{H}_S$, then it must be in factorized form $\implies$ initial factorization condition
- extra goody: we can apply all the tools recently developed for approximate recoverability (family is approximately steerable, family is approximately Markov, tripartite state is approximately Markov, etc)
✔ data-processing theorem: if there is a process, information always decreases
✔ reverse data-processing theorem: if information *always* decreases, then there exists a process
✔ data-processing inequality as a "physical principle": the flow of information determines the evolution
✔ analogy with strong (i.e., necessary and sufficient) second law-like statements (e.g., Lieb–Yngvasson formulation of adiabaticity)
✔ work in progress: applications to generalized resource theories

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