Quantum Information Processing in Non-Markovian Quantum Complex Systems

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Classical Markov chains: some nomenclature

Time convention: \( t_N \geq \cdots \geq t_1 \geq t_0 \).

- classical Markov chain:

\[
P(x_{t_N}, x_{t_{N-1}}, \ldots, x_{t_0}) = P(x_{t_N} | x_{t_{N-1}}) \cdots P(x_{t_1} | x_{t_0}) P(x_{t_0})
\]

- keywords: memorylessness, Markovianity, divisibility

- **physical divisibility** (Markov equation):

\[
P(x_{t_k}, x_{t_j}, x_{t_i}) = P(x_{t_k} | x_{t_j}) P(x_{t_j} | x_{t_i}) P(x_{t_i}), \text{ for any } k \geq j \geq i
\]

- **stochastic divisibility** (Chapman-Kolmogorov equation):

\[
P(x_{t_k} | x_{t_i}) = \sum_{x_{t_j}} P(x_{t_k} | x_{t_j}) P(x_{t_j} | x_{t_i}), \text{ for any } k \geq j \geq i
\]

\(\text{physical divisibility} \not\iff \text{stochastic divisibility}\)
The problem with quantum systems

Quantum stochastic processes are like sealed black boxes: an observation at time $t_1$ can “spoil” the process and any subsequent observation at later times $t_2 \geq t_1$.

Figure 1: Here $t_0$ is an initial time, at which the quantum system can be prepared (fully controlled). There is no direct quantum analogue of the $N$-time joint distribution $P(x_{t_N}, \ldots, x_{t_0})$. 
Quantum Dynamical Mappings

How to describe *quantum* stochastic processes then?

- time convention: \( t_N \geq \cdots \geq t_1 \geq t_0 \)
- open quantum systems formalism:
  \[
  \rho_S(t_i) := \text{Tr}_E \left\{ U_{t_0 \to t_i} \left[ \rho_S(0) \otimes \rho_E(0) \right] U_{t_0 \to t_i}^\dagger \right\}
  \]
- if the system is fully controlled at \( t_0 \), we obtain a sequence of CPTP linear maps by discarding the bath:
  \[
  \Phi_i(\cdot) := \text{Tr}_E \left\{ U_{t_0 \to t_i} \left[ \cdot \otimes \rho_E(0) \right] U_{t_0 \to t_i}^\dagger \right\}
  \]

**Definition**

A quantum dynamical mapping (QDM) is a sequence of CPTP linear maps \( (\Phi_i)_{0 \leq i \leq N} \) satisfying \( \Phi_0 = \text{id}_S \) (consistency condition).
Two approaches to quantum Markovianity

• **Global (extrinsic) picture**: Markovianity is a property of the whole system+bath compound (like, e.g., singular coupling regime, approximate factorizability, etc)

• **Reduced (intrinsic) picture**: Markovianity is a property of the resulting quantum dynamical mapping alone (like, e.g., information decrease, divisibility, etc)
A “Zoo” of Quantum Markovianities

A “Zoo” of Quantum Markovianities

Decreasing System Distinguishability (DSD)

- introduced in [Breuer, Laine, Piilo; PRL 2009], it provides the bridge between physical and information-theoretic Markovianity
- for any pair of possible initial states of the system, say, $\rho^1_S(0)$ and $\rho^2_S(0)$, consider the same pair evolved at later times $t_i > t_0$:

$$\rho^{1,2}_S(t_i) := \Phi_i \left[ \rho^{1,2}_S(0) \right]$$

- DSD condition:

$$\|\rho^1_S(t_i) - \rho^2_S(t_i)\|_1 \geq \|\rho^1_S(t_j) - \rho^2_S(t_j)\|_1, \quad \forall i \leq j$$

- interpretation: revival of distinguishability $\implies$ back-flow of information $\implies$ non-Markovianity
Divisibility (DIV)

- extends the idea of dynamical semigroups: $t \mapsto \Phi_t$ such that $\Phi_s \circ \Phi_t = \Phi_{t+s}$
- a QDM $(\Phi_i)_i$ is CPTP divisible if there exist CPTP linear maps $(\mathcal{E}_{i \rightarrow j})_{i \leq j}$, which we call propagators, such that $\Phi_j = \mathcal{E}_{i \rightarrow j} \circ \Phi_i$, for all $0 \leq i \leq j \leq N$

- DIV constitutes a quantum analogue of the Chapman-Kolmogorov equation (i.e., stochastic divisibility)
can we make these equivalent?
Strengthening DSD

- both **DSD** and **DIV** play an important role in information theory under the names of **data-processing inequality** and **degradability**, respectively
- **reverse data-processing theorems**: various generalizations of DSD that become equivalent to DIV (sometimes, however, bijectivity of all $\Phi_i$'s is required)

**A recent result (FB, 2018)**

Given a bipartite state $\omega_{RS}$, define its singlet fraction given $S$ as

$$
\mathcal{F}(\omega \mid S) := \sup_{\mathcal{D}:\text{CPTP}} \langle \Phi^+_{RS} \mid (\text{id}_R \otimes \mathcal{D}_S)(\omega_{RS}) \mid \Phi^+_{RS} \rangle.
$$

Denote $\omega_i := (\text{id}_R \otimes \Phi_i)(\omega_{RS})$. A QDM $(\Phi_i)_i$ satisfies DIV if and only if $\mathcal{F}(\omega_i \mid S) \geq \mathcal{F}(\omega_j \mid S)$, for all $j \geq i$ and all separable bipartite states $\omega_{RS}$. 
Visualizing the condition

• The thickness of the green lines depict the singlet fractions at any time:

$$\mathcal{F}(\omega_i|S) := \sup_{D:\text{CPTP}} \langle \Phi^+_{RS}|(\text{id}_R \otimes D_S \circ \Phi_i)(\omega_{RS})|\Phi^+_{RS} \rangle.$$

• A QDM $$(\Phi_i)_i$$ satisfies DIV iff $$\mathcal{F}(\omega_i|S) \geq \mathcal{F}(\omega_j|S)$$ for all initial separable states $$\omega_{RS}$$. 
Meaning of DIV

Why the propagators \((\mathcal{E}_{i \rightarrow j})_{i \leq j}\) are assumed to be CPTP?

Hence, CP-divisibility is equivalent to saying that the open evolution is “collisional,” in the sense that it can be realized by summoning a “fresh environment” at each time step.
To strengthen DSD or to relax DIV?

- But do the propagators \((\mathcal{E}_{i \to j})_{i \leq j}\) really need to be linear CPTP?
  - linearity is necessary (QDMs are linear)
  - trace-preservation (a linear constraint) also
  - instead, CP perhaps not: propagators could be just P or even less (e.g., statistical morphisms), and yet be related to important physical/computational/thermodynamical properties (like, e.g., the “locality” or “causality” of the evolution)

**A recent result (FB, 2018)**

A QDM \((\Phi_i)\) satisfies P-DIV if and only if \(\mathcal{F}(\omega_i|S) \geq \mathcal{F}(\omega_j|S)\), for all \(j \geq i\) and all **classical-quantum** bipartite states \(\omega_{RS}\).

**Remark.** Classical-quantum states have the form \(\omega_{RS} = \sum_k p_k |k\rangle \langle k|_R \otimes \omega^k_S\).
The QDM is **CP-divisible** iff $\mathcal{F}(\omega_i|S) \geq \mathcal{F}(\omega_j|S)$ for all initial separable states.

The QDM is **P-divisible** iff $\mathcal{F}(\omega_i|S) \geq \mathcal{F}(\omega_j|S)$ for all initial classical-quantum states.
Possible ideas in this direction

- to witness P-indivisibility, classical correlations are enough; for CP-indivisibility, separable non-classical states are required. Discord, anyone?
- it is known that CP-DIV can be decided by SDP: way to design efficient tests?
- to impose extra properties to DIV, e.g., thermality or group-covariance
- to understand P-DIV in a generalized circuit formalism (no extension possible, however no problem, because not in the black-box picture)
- relation to causality/time-locality? For example: can a-causal (time-nonlocal) processes arise in regimes of extreme non-Markovianity?
- to understand the information-theoretic and computational capabilities of such generalized circuit models, e.g., data-processing inequalities, computational/thermodynamical aspects, etc