Quantum Information Processing in Non-Markovian Quantum Complex Systems

Francesco Buscemi[1]

Nagoya–Freiburg Joint Project Kick-Off Meeting
Institute of Physics, Freiburg University, 14 May 2018

[1]Dept. of Mathematical Informatics, Nagoya University, buscemi@i.nagoya-u.ac.jp

Classical Markov chains: some nomenclature

Time convention: $t_N \geq \cdots \geq t_1 \geq t_0$.

- classical Markov chain:

$$P(\boldsymbol{x}_{t_N}, \boldsymbol{x}_{t_{N-1}}, \ldots, \boldsymbol{x}_{t_0}) = P(\boldsymbol{x}_{t_N}|\boldsymbol{x}_{t_{N-1}}) \cdots P(\boldsymbol{x}_{t_1}|\boldsymbol{x}_{t_0}) P(\boldsymbol{x}_{t_0})$$

- keywords: memorylessness, Markovianity, divisibility
- **physical divisibility** (Markov equation):
  $P(\boldsymbol{x}_{t_k}, \boldsymbol{x}_{t_j}, \boldsymbol{x}_{t_i}) = P(\boldsymbol{x}_{t_k}|\boldsymbol{x}_{t_j}) P(\boldsymbol{x}_{t_j}|\boldsymbol{x}_{t_i}) P(\boldsymbol{x}_{t_i})$, for any $k \geq j \geq i$
- **stochastic divisibility** (Chapman-Kolmogorov equation):
  $P(\boldsymbol{x}_{t_k}|\boldsymbol{x}_{t_i}) = \sum_{\boldsymbol{x}_{t_j}} P(\boldsymbol{x}_{t_k}|\boldsymbol{x}_{t_j}) P(\boldsymbol{x}_{t_j}|\boldsymbol{x}_{t_i})$, for any $k \geq j \geq i$

physical divisibility $\underset{\not\Longleftarrow}{\Longrightarrow}$ stochastic divisibility

The problem with quantum systems

Quantum stochastic processes are like sealed black boxes: an observation at time t_1 can "spoil" the process and any subsequent observation at later times $t_2 \geq t_1$.

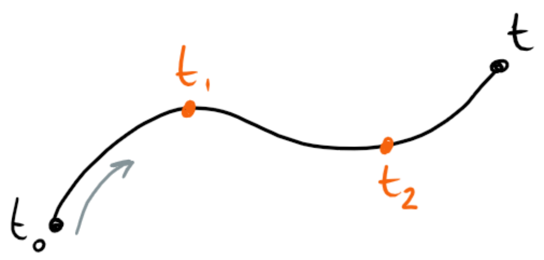


Figure 1: Here t_0 is an initial time, at which the quantum system can be prepared (fully controlled). There is no *direct* quantum analogue of the N-time joint distribution $P(\boldsymbol{x}_{t_N}, \ldots, \boldsymbol{x}_{t_0})$.

Quantum Dynamical Mappings

How to describe *quantum* stochastic processes then?

- time convention: $t_N \geq \cdots \geq t_1 \geq t_0$
- open quantum systems formalism:
  $\rho_S(t_i) := \mathrm{Tr}_E \left\{ U_{t_0 \to t_i} \left[\rho_S(0) \otimes \rho_E(0) \right] U^\dagger_{t_0 \to t_i} \right\}$
- if the system is fully controlled at t_0, we obtain a sequence of CPTP linear maps by discarding the bath:
  $\Phi_i(\cdot) := \mathrm{Tr}_E \left\{ U_{t_0 \to t_i} \left[\cdot \otimes \rho_E(0) \right] U^\dagger_{t_0 \to t_i} \right\}$

Definition

A quantum dynamical mapping (QDM) is a sequence of CPTP linear maps $(\Phi_i)_{0 \leq i \leq N}$ satisfying $\Phi_0 = \mathrm{id}_S$ (consistency condition).

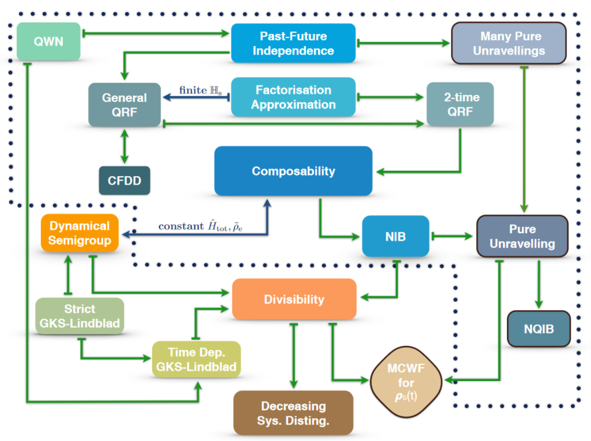
Two approaches to quantum Markovianity

- **Global (extrinsic) picture**: Markovianity is a property of the whole system+bath compound (like, e.g., singular coupling regime, approximate factorizability, etc)

- **Reduced (intrinsic) picture**: Markovianity is a property of the resulting quantum dynamical mapping alone (like, e.g., information decrease, divisibility, etc)

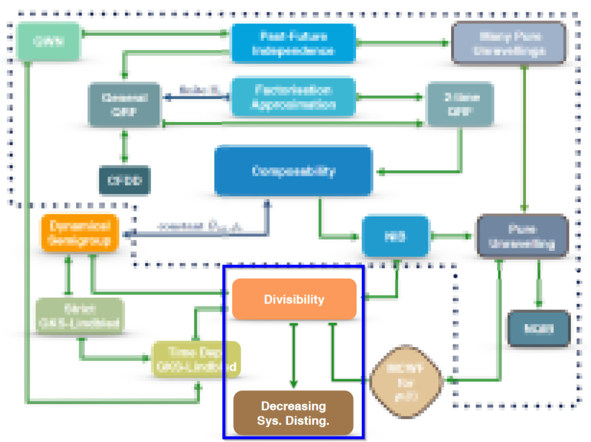
A “Zoo” of Quantum Markovianities

From: Li Li, Michael J. W. Hall, Howard M. Wiseman. *Concepts of quantum non-Markovianity: a hierarchy*. (arXiv:1712.08879 [quant-ph])



A “Zoo” of Quantum Markovianities

From: Li Li, Michael J. W. Hall, Howard M. Wiseman. *Concepts of quantum non-Markovianity: a hierarchy*. (arXiv:1712.08879 [quant-ph])



Decreasing System Distinguishability (DSD)

- introduced in [Breuer, Laine, Piilo; PRL 2009], it provides the bridge between physical and information-theoretic Markovianity
- for any pair of possible initial states of the system, say, $\rho_S^1(0)$ and $\rho_S^2(0)$, consider the same pair evolved at later times $t_i > t_0$:

$$\rho_S^{1,2}(t_i) := \Phi_i\left[\rho_S^{1,2}(0)\right]$$

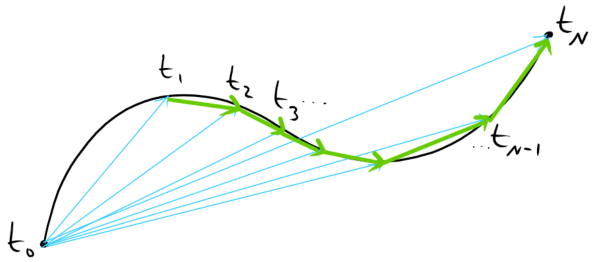
- DSD condition:

$$\|\rho_S^1(t_i) - \rho_S^2(t_i)\|_1 \geq \|\rho_S^1(t_j) - \rho_S^2(t_j)\|_1 \ , \quad \forall i \leq j$$

- **interpretation**: revival of distinguishability $\Longrightarrow$ back-flow of information $\Longrightarrow$ non-Markovianity

Divisibility (DIV)

- extends the idea of dynamical semigroups: $t \mapsto \Phi_t$ such that $\Phi_s \circ \Phi_t = \Phi_{t+s}$
- a QDM $(\Phi_i)_i$ is CPTP divisible if there exist CPTP linear maps $(\mathcal{E}_{i \to j})_{i \leq j}$, which we call propagators, such that $\Phi_j = \mathcal{E}_{i \to j} \circ \Phi_i$, for all $0 \leq i \leq j \leq N$



- DIV constitutes a quantum analogue of the Chapman-Kolmogorov equation (i.e., stochastic divisibility)

$$\mathrm{DIV} \underset{\not\Longleftarrow}{\Longrightarrow} \mathrm{DSD}$$

can we make these equivalent?

Strengthening DSD

- both **DSD** and **DIV** play an important role in information theory under the names of **data-processing inequality** and **degradability**, respectively
- **reverse data-processing theorems**: various generalizations of DSD that become equivalent to DIV (sometimes, however, bijectivity of all Φ_i's is required)

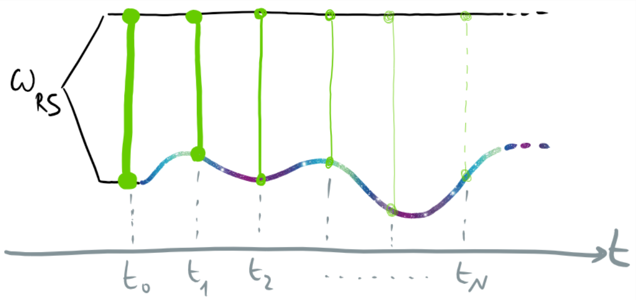
A recent result (FB, 2018)

Given a bipartite state ω_{RS}, define its singlet fraction given S as

$$\mathscr{F}(\omega|S) := \sup_{\mathcal{D}:\text{CPTP}} \langle\Phi^+_{RS}|(\text{id}_R \otimes \mathcal{D}_S)(\omega_{RS})|\Phi^+_{RS}\rangle \ .$$

Denote $\omega_i := (\text{id}_R \otimes \Phi_i)(\omega_{RS})$. A QDM $(\Phi_i)_i$ satisfies DIV if and only if $\mathscr{F}(\omega_i|S) \geq \mathscr{F}(\omega_j|S)$, for all $j \geq i$ and all **separable** bipartite states ω_{RS}.

Visualizing the condition



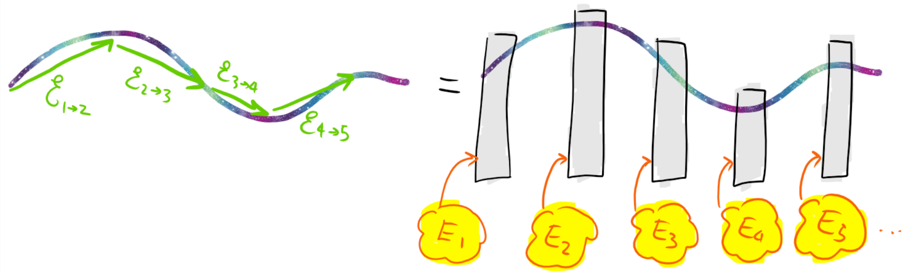
- The thickness of the green lines depict the singlet fractions at any time:

$$\mathscr{F}(\omega_i|S) := \sup_{\mathcal{D}:\text{CPTP}} \langle\Phi^+_{RS}|(\mathrm{id}_R \otimes \mathcal{D}_S \circ \Phi_i)(\omega_{RS})|\Phi^+_{RS}\rangle \, .$$

- A QDM $(\Phi_i)_i$ satisfies DIV iff $\mathscr{F}(\omega_i|S) \geq \mathscr{F}(\omega_j|S)$ for all initial **separable** states ω_{RS}.

Meaning of DIV

Why the propagators $(\mathcal{E}_{i\to j})_{i\leq j}$ are assumed to be CPTP?



Hence, CP-divisibility is equivalent to saying that the open evolution is "collisional," in the sense that it can be realized by summoning a "fresh environment" at each time step.

To strengthen DSD or to relax DIV?

- But do the propagators $(\mathcal{E}_{i\to j})_{i\leq j}$ *really* need to be linear CPTP?
  - linearity is necessary (QDMs are linear)
  - trace-preservation (a linear constraint) also
  - instead, CP perhaps not: propagators could be just P or even less (e.g., statistical morphisms), and yet be related to important physical/computational/thermodynamical properties (like, e.g., the "locality" or "causality" of the evolution)

A recent result (FB, 2018)

A QDM (Φ_i) satisfies P-DIV if and only if $\mathscr{F}(\omega_i|S) \geq \mathscr{F}(\omega_j|S)$, for all $j \geq i$ and all **classical-quantum** bipartite states ω_{RS}.

Remark. Classical-quantum states have the form $\omega_{RS} = \sum_k p_k |k\rangle\langle k|_R \otimes \omega_S^k$.

CP-DIV, P-DIV, and non-increasing singlet fractions

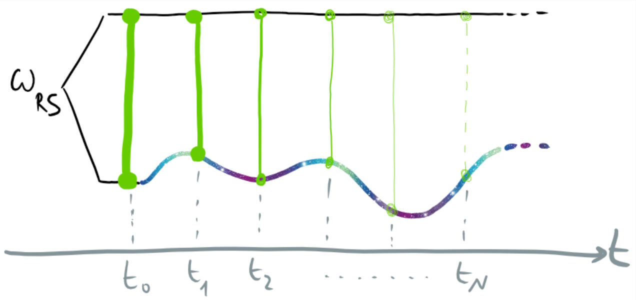


Figure 2: The varying thickness of the green lines depict the singlet fraction at any time.

- The QDM is **CP-divisible** iff $\mathscr{F}(\omega_i|S) \geq \mathscr{F}(\omega_j|S)$ for all initial **separable** states.
- The QDM is **P-divisible** iff $\mathscr{F}(\omega_i|S) \geq \mathscr{F}(\omega_j|S)$ for all initial **classical-quantum** states.

Possible ideas in this direction

- to witness P-indivisibility, *classical correlations* are enough; for CP-indivisibility, *separable non-classical* states are required. **Discord**, anyone?
- it is known that CP-DIV can be decided by SDP: way to design **efficient tests**?
- to impose extra properties to DIV, e.g., **thermality or group-covariance**
- to understand P-DIV in a **generalized circuit formalism** (no extension possible, however no problem, because not in the black-box picture)
- relation to **causality/time-locality**? For example: can a-causal (time-nonlocal) processes arise in regimes of extreme non-Markovianity?
- to understand the **information-theoretic and computational capabilities** of such generalized circuit models, e.g., data-processing inequalities, computational/thermodynamical aspects, etc