

When Is a Measurement Reversible?

Information Gain and Disturbance in Quantum Measurements (revisited)
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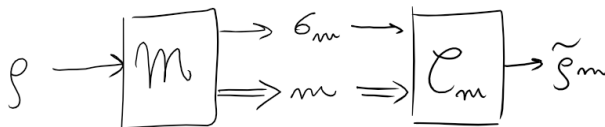
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- ✓ an **operation** transforming a quantum input (the density operator representing the state of the system before the measurement, i.e., the “pre-measurement state”) into a classical output (the “outcome” of the measurement) and a quantum output (the state of the system conditional on the observation of some outcome, i.e., the “post-measurement state”)
- ✓ mathematically, a measurement process is represented by a **completely positive instrument** (Ozawa, 1984), namely, a family $\mathfrak{M} = \{\mathcal{E}_m : m \in \mathcal{M}\}$ of completely positive maps $\mathcal{E}_m$, indexed by the outcome set $\mathcal{M} = \{m\}$, and such that $\sum_m \mathcal{E}_m$ is trace-preserving (completeness relation)
- ✓ upon input ρ , the outcome m is obtained with probability $p(m) = \text{Tr}[\mathcal{E}_m(\rho)]$ and the corresponding post-measurement state is $\sigma_m = \mathcal{E}_m(\rho)/p(m)$

Measurement Reversibility

- ✓ information extraction requires measurement, but measurements occur by interaction with a probe, and interactions cause disturbance $\Rightarrow$ **information–disturbance tradeoff** in quantum measurements
- ✓ the intuition is clear, less so the mathematical formulation (state dependent/independent, figures of merit, average/single-outcome, with/without feedback, etc); in particular, are measurements always irreversible?
- ✓ **correction scheme**:



where $\{\mathcal{C}_m : m \in \mathcal{M}\}$ is a family of error-correcting CPTP maps (independent of ρ)

- ✓ **average disturbance**: “distance” between ρ and $\tilde{\rho} = \sum_m p(m) \tilde{\rho}_m = \sum_m (\mathcal{C}_m \circ \mathcal{E}_m)(\rho)$
- ✓ **Fact.** Among all measurements with the same outcome statistics, **efficient measurements** (i.e., $\mathcal{E}_m(\cdot) = E_m \cdot E_m^\dagger$) are the least disturbing one: we will hence focus on these only



- ✓ the **Groenewold-Lindblad-Ozawa information gain** is defined as the average entropy decrease due to measurement: $I_G(\rho, \mathfrak{M}) \triangleq H(\rho) - \sum_m p(m)H(\sigma_m)$
- ✓ for efficient measurements (i.e., $\sigma_m \propto E_m \rho E_m^\dagger$), $I_G(\rho, \mathfrak{M})$ is always non-negative and equals the **optimal rate of measurement compression**: minimum rate at which classical information needs to be sent in order to communicate the measurement outcome to a receiver (Winter, 2004)
- ✓ $I_G(\rho, \mathfrak{M})$ quantifies intrinsic/useful data; the difference $H(M) - I_G(\rho, \mathfrak{M})$ quantifies extrinsic/useless data
- ✓ tantalizing analogy with **heat and work in thermodynamics**

The Technical Lemma

Theorem

Let Φ be a subunital (i.e., $\Phi(I) \leq I$) CPTP map. Then there exists a CPTP map Ψ such that, for any ρ , $H[\Phi(\rho)] - H(\rho) \geq D[\rho \| (\Psi \circ \Phi)(\rho)]$.

Proof.

- 1 Φ is CPTP subunital map $\iff \Phi^\dagger$ is CP trace-non-increasing unital map
- 2 $H[\Phi(\rho)] - H(\rho) = \text{Tr}[\rho \log \rho] - \text{Tr}[\Phi(\rho) \log \Phi(\rho)] = \text{Tr}[\rho \log \rho] - \text{Tr}[\rho \Phi^\dagger(\log \Phi(\rho))] \geq \text{Tr}[\rho \log \rho] - \text{Tr}[\rho \log\{(\Phi^\dagger \circ \Phi)(\rho)\}] = D[\rho \| (\Phi^\dagger \circ \Phi)(\rho)]$, where inequality follows from operator concavity of the logarithm and the operator Jensen inequality for positive unital maps
- 3 $\Psi(\cdot) \triangleq \Phi^\dagger(\cdot) + \text{Tr}[(\text{id} - \Phi^\dagger)(\cdot)] \omega$, where ω is arbitrary but fixed density operator
- 4 since $\Psi(\cdot) \geq \Phi^\dagger(\cdot)$, it follows that $D[\rho \| (\Phi^\dagger \circ \Phi)(\rho)] \geq D[\rho \| (\Psi \circ \Phi)(\rho)]$



Stronger than any other (currently known) recoverability results:

- 1 the lemma holds also for positive (not necessarily completely positive) Φ
- 2 the explicit form of correction Ψ is known
- 3 $D(\rho \| \sigma) \geq D_{\mathbb{M}}(\rho \| \sigma) \geq -\log F(\rho, \sigma)$, where $F(\rho, \sigma) \triangleq \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$

- ✓ given an efficient measurement process $\mathfrak{M} = \{E_m\}$, construct the CPTP map $\mathcal{M}(\cdot) = \sum_m E_m \cdot E_m^\dagger \otimes |m\rangle\langle m|$

- ✓ then

$$\mathcal{M}(I) = \sum_m E_m E_m^\dagger \otimes |m\rangle\langle m| = \begin{pmatrix} E_1 E_1^\dagger & 0 & 0 & \cdots \\ 0 & E_2 E_2^\dagger & 0 & \cdots \\ 0 & 0 & E_3 E_3^\dagger & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- ✓ $\sum_m E_m^\dagger E_m = I \implies E_m^\dagger E_m \leq I$ for all $m \implies E_m E_m^\dagger \leq I$ for all $m \implies \mathcal{M}(I) \leq I \otimes I$
- ✓ in other words, the channel $\mathcal{M}$ is subunitary and the Technical Lemma can be applied

Second Law for Efficient Quantum Measurements

- ✓ recall the definition: $\mathcal{M}(\rho) = \sum_m E_m \rho E_m^\dagger \otimes |m\rangle\langle m|$
- ✓ the Technical Lemma implies that there exists a correction scheme $\{\mathcal{C}_m\}$ such that $H[\mathcal{M}(\rho)] - H(\rho) \geq D[\rho \| \sum_m (\mathcal{C}_m \circ \mathcal{E}_m)(\rho)]$ for all ρ
- ✓ $H[\mathcal{M}(\rho)] - H(\rho) = H(M) - [H(\rho) - \sum_m p(m)H(\sigma_m)] = H(M) - I_G(\rho, \mathfrak{M})$

Theorem (Strengthened Second Law for Efficient Quantum Measurements)

Let $\mathfrak{M} = \{E_m\}$ be an efficient quantum measurement process. Then there exists a correction scheme $\{\mathcal{C}_m\}$ such that

$$H(M) - I_G(\rho, \mathfrak{M}) \geq D\left(\rho \left\| \sum_m (\mathcal{C}_m \circ \mathcal{E}_m)(\rho)\right.\right),$$

for all ρ .

- ✓ hence, **extrinsic noise measures the irreversibility of efficient measurements** (as does entropy for adiabatic processes)
- ✓ indeed, in [K. Jacobs, PRA **80**, 012322 (2009)] the inequality $H(M) - I_G(\rho, \mathfrak{M}) \geq 0$ is interpreted as the **second law for quantum measurements**; the above Theorem considerably strengthens this



Thank You

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