Quantum Network Coding – How can network coding be applied to quantum information?

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***Abstract*—Nowadays, network coding has been a very successful topic in information theory including many applications such as wireless networks while one of the simplest benefits is the efficient transmission in networks by allowing us to encode the transmitted information at every intermediate node. Since quantum information is much more expensive for communication than classical information, it was natural that this benefit made quantum researchers motivate the study of "quantum network coding." This paper reports the current status of quantum network coding. At present, quantum network coding mostly means sending quantum information (rather than classical information) in a quantum network. Since quantum information cannot be cloned (the quantum no-cloning theorem), multiple unicast networks have been well-studied (in several settings). We present some of the known possibilities and limitations, and future works of quantum network coding, focusing on multiple unicast networks.**

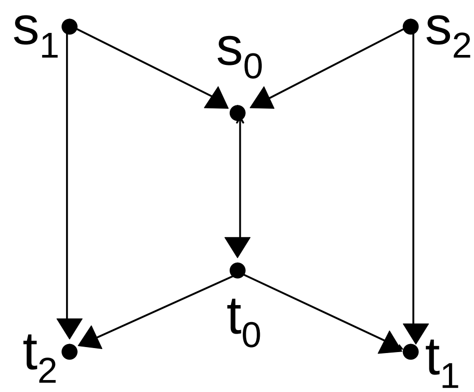

Fig. 1. Butterfly network

I. Introduction

So far, many applications on communication systems have been proposed using quantum information. For instance, the Bennett-Brassard quantum key distribution system (BB84) [3] provides us an information-theoretically secure key distribution system, which is impossible by using only classical information. The study of quantum communication systems have been also extended from point-to-point communication channels to networks. On the contrary, sending quantum information faithfully in those systems is basically not so easy since it is quite weak for interactions with an environment system. Therefore, for physically implementing those systems, it is very important to reduce the amount of quantum communication as less as possible.

For this purpose, it is a natural consideration to apply the idea of network coding [1] to quantum networks. The study of quantum network coding was initiated by Hayashi et al. [12] for the butterfly network (Fig. 1), a typical example to illustrate the power of network coding, where source nodes s_1 and s_2 want to send their quantum states to their corresponding target nodes t_1 and t_2, respectively, where the capacity of each edge is one. A reason why this network was chosen as the starting point of the study was that it is a multiple unicast network. In the theory of network coding, multicast networks were rather well-studied in the early stage [1], [22], [15], while in the quantum setting, the no-cloning theorem [27] prohibits us from multicasting a quantum state.

In this paper, we give a short survey on the known results of quantum network coding, mainly focusing on the multiple unicast networks. Section II provides a number of basics on quantum information for reading the later sections. In Section III, we report the results on quantum network coding in the most basic setting where we can send a single qubit for each edge in the underlying networks. Sections IV and V include the studies for the case where additional resources such as classical communication or entanglement are available with the underlying quantum network. In Section VI, we mention the results on the multicast networks. Finally, we give a concluding remark in Section VII.

II. Basics on Quantum Information

We review a minimum of notations and facts on quantum information. For more details, see standard textbooks such as Nielsen and Chuang [23].

Quantum states. Mathematically, a quantum system is represented by a complex Hilbert space, and a (pure) quantum state on the system is represented by a unit vector in the Hilbert space. In particular, a quantum bit (called *qubit*) is a unit vector $|\psi\rangle = a|0\rangle + b|1\rangle$ in the 2-dimensional space $\mathcal{H}_2$ spanned by the orthonormal basis $|0\rangle$ and $|1\rangle$ (which correspond to classical bits 0 and 1, respectively). Then, an m-qubit state is a unit vector in the space $\mathcal{H}_2^{\otimes n}$ (the m tensor products of $\mathcal{H}_2$).

For instance,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

is a 2-qubit state. Note that $|\Phi^+\rangle$ cannot be written as a tensor product of 1-qubit states on the first and second subsystems, that is, as a vector in the form of $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$. We say that such a state is *entangled* between the two subsystems. In particular, $|\Phi^+\rangle$ is called an *EPR (Einstein-Podolsky-Rosen) pair*, which is a very useful entangled state for quantum information processing, as seen below.

One of the measures for closeness between two quantum states $|\psi\rangle$ and $|\phi\rangle$ is the *fidelity*, which is defined to be the absolute value of their inner product. Note that if the fidelity is one, then $|\psi\rangle$ and $|\phi\rangle$ represent the same state.

Quantum operation. A quantum operation is represented by a map $\mathcal{M}$ that transforms an input state $|\psi\rangle$ to an output state $\mathcal{M}(|\psi\rangle)$, which may be generally a probability distribution of quantum states (called a mixed state). Operationally, any quantum operation can be represented by the following three steps: (i) Add an additional system that is prepared to a fixed state like $|0\rangle$; (ii) Apply a unitary transformation on the composite system consisting of the input system and the additional system; (iii) Take a part (or the whole) of the total system as the output. The no-cloning theorem [27] says that for any d-dimensional space $\mathcal{H}_d$ with $d \geq 2$, there is no quantum operation that maps any state $|\psi\rangle$ in $\mathcal{H}_d$ to $|\psi\rangle \otimes |\psi\rangle$.

In general, an m-qubit (noisy) channel is a quantum operation that transforms m-qubit states to another m-qubit (mixed) states, while this paper considers only the trivial channel with unit capacity, that is, the channel outputting an input single-qubit state without any change (in other words, we consider only noiseless quantum channels).

Quantum teleportation and dense coding. The quantum teleportation [4] and (super-)dense coding [5] are standard communication protocols in quantum information processing. In both protocols, we assume that a party A and another party B share an EPR pair $|\Phi^+\rangle$ (that is, A has the first subsystem of $|\Phi^+\rangle$ and B has the second subsystem). Then, the *quantum teleportation* is a quantum operation that can send a single qubit state $|\psi\rangle$, which is in a system different from $|\Phi^+\rangle$, owned by A (who does not know what the state is) into B by sending two classical bits from A to B (and then the EPR pair is consumed). This means that we can substitute sending two bits and sharing one EPR pair (between the sender and the receiver) for sending one qubit. In a complementary style, *dense coding* is a quantum operation that can send two classical bits by sending only one qubit (and the shared EPR pair is also consumed). This means that dense coding substitutes sending one qubit and sharing one EPR pair for sending two bits.

III. Results on Basic Setting

Now we provide a basic setting of quantum network coding in a multiple unicast network, which was introduced in Ref. [12]. A (multiple unicast) quantum network N is represented by a directed acyclic graph with source nodes $s_1, \ldots, s_k$ and target nodes $t_1, \ldots, t_k$, where every edge of N represents a quantum channel with unit capacity. Each source node s_i has a single-qubit state $|\psi_i\rangle$. We are allowed to apply any quantum operation at every node v: when v has l incoming edges and m outgoing edges, it can apply any quantum operation $\mathcal{M}_v$ that maps an l-qubit state coming from the l incoming edges into an m-qubit state, which is sent to the m outgoing edges. We say that N is *quantumly solvable with fidelity* F if there is a choice of quantum operations $\mathcal{M}_v$ (called a solution) such that every $|\psi_i\rangle$ can be sent from s_i to t_i with fidelity at least F (where the fidelity between $|\psi_i\rangle$ and the mixed state at t_i is the average of the fidelities between $|\psi_i\rangle$ and pure states in the mixed state). In particular, when $F = 1$, it is simply called quantumly solvable. This can be regarded as a natural quantum analogue of the corresponding classical multiple unicast network where the channels, sources, and operations at every node are all classical.

As is well-known, the butterfly network is classically solvable, which is illustrated in Fig. 2. A natural idea for giving a solution in the quantum setting is to simulate the classical solution, while we find immediately that there are nontrivial points for doing that. The first point is in the operations at the source nodes s_1, s_2 and node t_0, which duplicate the information. As mentioned already, the no-cloning theorem exists in the quantum world, which makes it impossible to follow the same strategy. The second point is in the operations at nodes s_0, t_1, t_2, which take the XOR of the two incoming bits. For this point, it is difficult even to imagine a quantum analogue of this operation properly. Intuitively, a qubit state $a|0\rangle + b|1\rangle$ is a continuous object (a, b can take any values satisfying $|a|^2 + |b|^2 = 1$), and hence it may seem to be the liquid flow rather than the information flow. However, converting this intuition into an impossibility proof is not so easy since it is allowed to apply any quantum operation at six nodes, which may induce entanglement among the nodes and then the analysis becomes quite hard. Hayashi et al. [12] considered a special case where the source at node s_2 is a classical bit in order to make the analysis mild, and showed that even for this case, the butterfly network is not quantumly solvable with fidelity 0.983, using a geometric view of qubits (called the Bloch ball [23]).

The above basic setting by Ref. [12] is the so-called one-shot, that is, one qubit at each source node must be sent to the corresponding target node by a single use of the network. Leung, Oppenheim and Winter [20] extended this setting to the following asymptotic version. We say that a *rate* $(r_1, \ldots, r_k)$ *is achievable in a quantum network* N if there is a choice of quantum operations such that by n uses of N, each s_i can send $n(r_i - \delta_n)$ qubits to t_i with fidelity $1 - \epsilon_n$, where $\delta_n, \epsilon_n \to 0$ as $n \to \infty$. In this asymptotic setting, Ref. [20] investigated inner and outer bounds of the rates in several simple networks. In the butterfly network, it was proven that the rate region was bounded by $r_1 + r_2 \leq 1$, which is trivially achievable by routing. In their proof, any protocol on the butterfly network was reduced to a quantum secret sharing protocol [8] where the

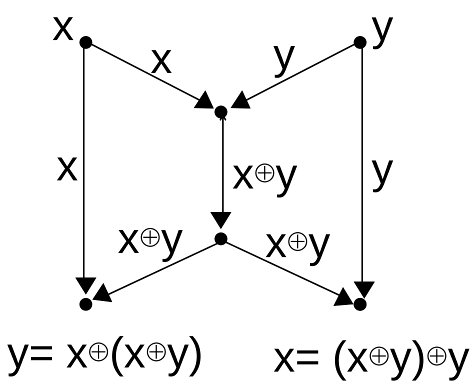

Fig. 2. Solution for the butterfly network

quantum secret is the two source qubits. Then, they gave the above outer bound by applying a lower bound on the quantum secret sharing [9], [14]. After the first version of Ref. [20], Hayashi [11] also proved a similar impossibility result without reducing to the quantum secret sharing by using information-theoretic arguments more directly. He also improved the upper bound of the fidelity for the one-shot case in Ref. [12] to 0.951.

One may expect from the negative results for the butterfly network that the optimal rates in all quantum networks are achievable by routing. However, Jain, Franceschetti and Meyer [16] observed that there exists a quantum network such that the achievable rate by network coding is k times the rate by routing (where k is the number of source-target pairs). This example was constructed based on the classical example in Ref. [10] by using quantum teleportation and dense coding, which allow us to take advantage of directed edges that are trivially useless by any routing protocol.

IV. With Free Classical Communication

In this section, we give an overview of the studies for the case where classical communication is available in addition to the basic quantum networks. This setting can be considered as the second-best when quantum network coding is impossible in the basic setting since the cost of classical communication is much cheaper than that of quantum communication.

A. Free Two-way Classical Communication

First, we consider the case where classical communication is freely available between any two nodes (which is equivalent to the case where it is only allowed between neighboring nodes of the underlying graph as long as the graph is connected). In this case, Leung et al. [20] made an important observation: *the underlying quantum network becomes undirected.* In fact, we can send a qubit in the reverse direction of each directed edge by first preparing an EPR pair using the directed quantum channel corresponding to the edge, and then by applying quantum teleportation using two free classical bits and the EPR pair. For the butterfly network, this enables us to send two qubits from s_1 to t_1 by a single use of the network, and two qubits from s_2 to t_2 by another single use. Thus, the rate $(r_1, r_2) = (x, 2-x)$ (where $0 \le x \le 2$) becomes achievable by time sharing (and this is shown to be optimal by a simple min-cut argument).

On the contrary, Kobayashi et al. [17], [19] showed the following relation between classical and quantum network coding in general multiple unicast networks.

Theorem 1. *If the rate $(r_1, \ldots, r_k)$ is achievable in a classical network, then the same rate is also achievable in the corresponding quantum network under free classical communication.*

To show this, they constructed a quantum protocol which, given a classical protocol, simulates faithfully the operations in this classical protocol at each node (recall that such a simulation is impossible without free classical communication, as described in Section III). Thus, if the classical protocol is done in the one-shot setting, their quantum protocol can be also done in the one-shot setting.

We notice that the converse of Theorem 1 is trivially false when the classical network is directed since the quantum network becomes undirected due to free classical communication. For instance, the graph consisting of only one directed edge (s_1, t_1) is a simple counter-example. However, if the classical network is undirected, it is open to show whether the converse holds or not. (This question might be difficult since the power of classical network coding in the undirected case has not yet been well understood [21].)

B. Free One-way Classical Communication

Leung et al. [20] studied the case where classical communication is freely available according to the directed edges of the underlying graph. In this case, we cannot reverse the edges at will. But we can increase the rates in some networks, compared to the case of no additional resources. For instance, the rate $(r_1, r_2) = (0.5, 1)$ is achievable in the butterfly network as follows: (i) s_1 sends the two subsystems of an EPR pair to s_0 and t_2, respectively. (ii) s_2 sends s_0 a source qubit, and s_0 teleports it to t_2 using the EPR pair and free two bits (which can be sent via the directed path $s_0 \to t_0 \to t_2$). (iii) s_1 and s_2 send their qubits by routing. This protocol uses the network twice while one qubit is sent from s_1 to t_1, and two qubits are sent from s_2 to t_2. A similar protocol with time-sharing achieves the rate region $\{(r_1, r_2) \mid r_1, r_2 \le 1,\ r_1 + r_2 \le 1.5\}$, which was proven to be optimal.

Moreover, they also studied the case where classical communication is available in the reverse direction to the edges in quantum networks.

V. With Free Entanglement

This section deals with the case where entanglement is allowed as additional resources. While entanglement is not cheaper than classical communication, there is an advantage that we can prepare it offline, that is, at any time. This type of studies is also motivated by the quantum information-theoretic significance that investigates the power of entanglement in quantum networks.

A. Entanglement between Any Two Nodes

In this case, any two nodes in a quantum network share any entangled state at will. Leung et al. [20] observed two facts that can be immediately obtained from quantum teleportation and dense coding. The first fact is the exact relation between the amounts of quantum and classical communication that can be sent on a quantum network.

Proposition 1. *Under free entanglement, the achievable rate for "quantum communication" in a quantum network is exactly half of that for "classical communication" in the same network.*

Proof. First, we assume that a rate $(r_1, \ldots, r_k)$ for quantum communication is achievable in a quantum network N. This means that $n(r_i - \delta_n)$ qubits can be sent between any pair (s_i, t_i) by n uses of N. Noting that they can use shared EPR pairs from their free entanglement, $2n(r_i - \delta_n)$ bits can be sent between them by n uses of N by virtue of dense coding.

Conversely, assume that a rate $(r_1, \ldots, r_k)$ for classical communication is achievable. This implies that $n(r_i - \delta_n)$ bits can be sent between any pair (s_i, t_i) by n uses of N. Then, $\frac{1}{2}n(r_i - \delta_n)$ qubits can be sent between them using their shard EPR pairs by virtue of quantum teleportation. □

Leung et al. gave the exact rate region $\{(r_1, r_2) \mid r_1, r_2 \leq 2\}$ for classical communication in the butterfly network. By Proposition 1, this implies that the rate region for quantum communication is $\{(r_1, r_2) \mid r_1, r_2 \leq 1\}$.

The second fact is a relation between the amount of quantum communication on a quantum network and the amount of classical communication on the corresponding classical network.

Proposition 2. *The achievable rate for quantum communication in a quantum network under free entanglement is at least that for classical communication in the corresponding classical network.*

Proof. Assuming that a rate $(r_1, \ldots, r_k)$ for classical communication is achievable, this implies that $n(r_i - \delta_n)$ bits can be sent between any pair (s_i, t_i) by n uses of the classical network. Noting that any pair (s_i, t_i) shares EPR pairs in the quantum case, it suffices to send $2n(r_i - \delta_n)$ bits from s_i to t_i for applying quantum teleportation. In fact, we can do that by applying dense coding using an EPR pair between each two neighboring nodes and one qubit which is allowed to send by the edge connecting the two nodes. □

In Ref. [20], the converse of Proposition 2 was conjectured (for instance, the converse holds in the butterfly network), but it still remains an interesting open question. If the conjecture is true, it implies that by Proposition 1, the rates for classical communication in quantum networks (even with free entanglement) is at most twice as much as those in classical networks, which extends the known results for point-to-point communication channels [13], [7] to networks.

B. Entanglement between Neighboring Nodes

In this case, any two neighboring nodes are allowed to share entanglement. In fact, the Hayashi's impossibility proof [11] mentioned in Section III implies that the achievable rate region in the butterfly network is also the same as that for the case of no additional resources.

Recently, motivated by quantum repeater networks [6], Satoh, Le Gall and Imai [24] studied the setting where any two neighboring nodes share EPR pairs and free classical communication is allowed, but no quantum communication is available and any extra qubits other than receiving qubits are not allowed to use at each node (which make the physical implementation easier). In this setting, they gave a protocol for the butterfly network that can send two source qubits simultaneously by a single use of the network.

C. Entanglement between Source Nodes

In this case, any source nodes are allowed to share entanglement. Unfortunately, for even the butterfly network, the exact rate region remains open (as far as the author knows). Hayashi [11] introduced a bit flexible setting where each edge can choose sending one qubit or two bits. (This was motivated by the equivalence between one qubit and two bits under shared entanglement via quantum teleportation and dense coding.) Then, he showed that two source qubits can be sent simultaneously by a single use of the network. This possibility result can be regarded as swapping two source qubits on the butterfly network. Under this viewpoint, Soeda et al. [26] investigated which two-qubit operations can be done on the butterfly network (it can be regarded as a quantum analogue of the study of network computing [2]).

VI. Results on Multicast Networks

As said before, by the no-cloning theorem, we cannot follow many beautiful results on classical network coding for the multicast networks in a direct analogue. However, there are several works by restricting quantum sources or changing the task. Shi and Soljanin [25] considered the situation that each source node s_i has k copies of a state, $|\psi_i\rangle^{\otimes k}$, and wants to send one copy $|\psi_i\rangle$ to each of k target nodes, and constructed a protocol for sending them simultaneously using lossless quantum compression of the copies. Kobayashi et al. [18], [19] considered the task that shares the so-called (Schrödinger's) cat state $\frac{1}{\sqrt{2}}(|0\rangle^{\otimes(k+1)} + |1\rangle^{\otimes(k+1)})$ (which is a generalization of the EPR pair) among each of sources and k target nodes, and constructed an efficient quantum protocol for this task, based on the corresponding classical multicasting protocol.

VII. Concluding Remarks

We summarize the achievable rate region in the butterfly network for quantum communication in Table I, where N, C1, C2, E1, and E2 represent the basic settings with no additional resources, with free classical communication among any nodes, with free classical communication according to the directed edges, with free entanglement among any two nodes, and with free entanglement between neighboring nodes,

N	$\{(r_1, r_2) \mid r_1 + r_2 \leq 1\}$
C1	$\{(r_1, r_2) \mid r_1 + r_2 \leq 2\}$
C2	$\{(r_1, r_2) \mid r_1, r_2 \leq 1,\ r_1 + r_2 \leq 1.5\}$
E1	$\{(r_1, r_2) \mid r_1, r_2 \leq 1\}$
E2	$\{(r_1, r_2) \mid r_1 + r_2 \leq 1\}$

TABLE I

respectively. The study of quantum network coding has not yet been much developed. In particular, there are very few results on general networks for several difficulties, for example, analyzing possible entanglement among the nodes in the networks to show the outer bound. We hope that further new techniques and results on quantum network coding for general graphs would appear in the near future.

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