Quantum Network Coding and the Current Status of Its Studies

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***Abstract*—Considering extensive studies on network coding and quantum information theory in this century, it was just a matter of time that the study of quantum network coding was initiated. In some sense, it is rather important to ask if the amount of quantum bits communicated on a network can be reduced by allowing encoding at intermediate nodes on the network since quantum communication is much more expensive than classical communication in principle. The first result by Hayashi et al. in 2006 was basically negative for this question – quantum network coding does not give any benefit for reducing the amount of qubits sent on the butterfly network. However, this was just the first step to explore the theory of quantum network coding. After their work, it has been shown that the amount of communicated qubits can be reduced in several changes of the settings such as by allowing extra classical communication, while there has been further negative results. In this paper, we review the basic concept of quantum network coding, and report the currently known positive and negative results on quantum network coding in various settings.**

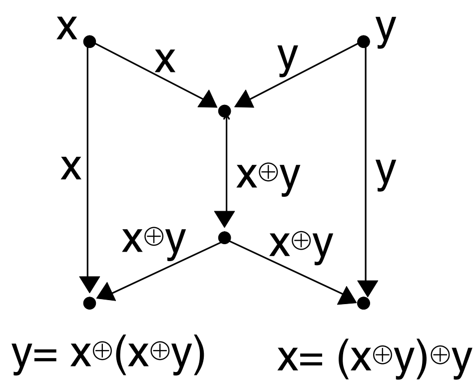

Fig. 1. Network coding on the butterfly network: the two source nodes are s_1 (the upper-left node) and s_2 (the upper-right node) and the two target nodes are t_1 (the lower-right node) and t_2 (the lower-left node).

I. Introduction

For better communication over a network, we may encode the information at intermediate nodes on the network – this is a conceptual message of network coding [1]. Among many advantages by network coding [4], the most fundamental one is to reduce the amount of communication over the underlying network. Therefore, it seems to be expected that the first motivation for importing the idea of network coding into quantum information theory was to explore the possibility that the amount of quantum communication can be reduced.

In 2006, Hayashi et al. [8] initiated the study of *quantum network coding* for the butterfly network, the most famous network in network coding theory. Figure 1 depicts the butterfly network and the well-known classical network coding for sending two bits x and y from the two source nodes s_1 and s_2 (the upper-left and upper-right nodes) to the two target nodes t_1 and t_2 (the lower-right and lower-left nodes), respectively, under the restriction that all the capacities of the edges are one. The inevasible question made in Ref. [8] was whether the quantum analogue is possible or not. That is, can we send two unknown qubits (quantum bits) $|\psi_1\rangle$ and $|\psi_2\rangle$ under the restriction that all the edges can send only single qubits (the quantum capacities are one) by devising quantum operations at nodes? Here it may be meaningful to notice that the chosen network is not a multicast network (deeply explored in classical network coding theory [1], [18], [12]), but a multiple unicast network (less understood relatively despite of a number of partial results [24], [11], [9]). This is because it is impossible in principle to multicast an unknown quantum state due to the no-cloning theorem[1].

The difficulty of quantum network coding already appears well in the example of the butterfly network when we try to mimic the classical network coding. First, in classical network coding, the three nodes with two outgoing edges duplicate the information while, in the quantum case, the no-cloning theorem prevents doing it. Second, in the classical case, the other three nodes take the exclusive-or ($\oplus$ in Figure 1) of the two received bits while it seems quite difficult to consider its quantum version in a suitable way. In Ref. [8], it was shown that two qubits cannot be sent faithfully on the butterfly network (while it was also shown that they can be sent with a nontrivial fidelity). This result was negative for the possibility of quantum network coding while it also opened the door for extensive researches in this topic to seek the possibility and the limit of quantum network coding, including various settings where extra resources are available. This paper briefly reports currently known results on quantum network coding with several open problems. In what follows, we skip the basics

[1]This consideration is simply taken in principle. In fact, Shi and Soljanin in 2006 [22] studied multicasting in a quantum network under the assumption that the source nodes have many copies of a quantum state to be sent, in order to prevent the no-cloning theorem. In this case we can use a lossless compression at source nodes, rather than intermediate nodes, due to the structure of the symmetric subspace in which many copies live.

of quantum information including quantum teleportation and dense coding, which are referred to textbooks on quantum information [6], [19], [25].

II. Preliminaries

First, we present the notion of the quantum solvability on the quantum network coding [8]. A quantum network[2] N is represented by a directed acyclic graph with source nodes $s_1, \dots, s_k$ and target nodes $t_1, \dots, t_k$, where every edge in N represents a quantum channel with unit capacity. Each source node s_i has a single-qubit state $|\psi_i\rangle$. Any quantum operation (including non-unitary operations) is allowed to apply at every node v: when v has l incoming edges and m outgoing edges, it can apply any quantum operation $\mathcal{M}_v$ that maps an l-qubit state coming from the l incoming edges into an m-qubit state, which is sent to the m outgoing edges. We say that N is *quantumly solvable with fidelity* F if there is a choice of quantum operations $\mathcal{M}_v$ (called a solution) such that every $|\psi_i\rangle$ can be sent from s_i to t_i with fidelity at least F (where the fidelity between $|\psi_i\rangle$ and the state output at t_i is considered). In particular, when $F = 1$, it is simply called quantumly solvable. The above notion is described for the case where the quantum state is a qubit state (two-dimensional state) while it may be naturally extended to qudit states (d-dimensional states). Another notice is that the k sources are independent states in the above definition while many protocols in the literatures (such as Ref. [7], [14], [16]) work even when these are entangled.

Second, we present the notion of achievable rates by quantum network coding [17], which is asymptotic and may be a stronger notion to state the impossibility results when the fidelity approaches to 1. Consider a quantum network N with k pairs of source-target nodes $(s_1, t_1), \dots, (s_k, t_k)$. A *rate* $(r_1, \dots, r_k)$ *is achievable in* N if there is a choice of quantum operations such that by n uses of N, each s_i can send $n(r_i - \delta_n)$ qubits to t_i with fidelity $1 - \epsilon_n$, where $\delta_n, \epsilon_n \to 0$ as $n \to \infty$. This notion allows us to use a more flexible protocol such as time-sharing.

III. Fundamental limits on quantum network coding

The difference between classical and quantum network codings can be typically seen in the butterfly network. As mentioned in the introduction, Ref. [8] gave the first negative result on the possibility of quantum network coding. Precisely, it was shown that the butterfly network is not quantumly solvable with fidelity 0.983. However, this result was obtained by using a geometric view specific to qubit states, and hence it does not deny the possibility of being quantumly solvable in the asymptotic sense, that is, achieving rate $(1, 1)$ in the butterfly network. Such an impossibility result was shown by Leung et al. [17]; the region of the achievable rate (r_1, r_2) in the butterfly network is bounded by $r_1 + r_2 \leq 1$. In fact, this is the optimal since any rate (r_1, r_2) satisfying $r_1 + r_2 = 1$ can be achievable by mixing two simple routing strategies (sending $|\psi_1\rangle$ through the s_1-t_1 path and sending $|\psi_2\rangle$ through the s_2-t_2 path) with time sharing. Therefore, the result in Ref. [17] says that the optimal rate is achievable without any coding, and hence quantum network coding is useless in the butterfly network (as long as the time sharing is allowed), which contrasts with the classical case. The proof of Ref. [17] was based on a reduction to some limit of quantum secret sharing [10]. Later, Hayashi [7] refined their proof and gave a more direct proof which does not use quantum secret sharing. This proof also enabled him to improve the upper bound on the fidelity of quantum solvability in Ref. [8] to 0.951 (though this upper bound is still far from the lower bound obtained in Ref. [8], which is approximately 0.52), and furthermore the proof can be applicable to extended settings (as seen in Section IV-B).

For other networks, there are only a few results at the current stage. Ref. [17] extended the arguments for their outer bound to a shallow network with three source-target pairs (called the inverted crown network), which appeared previously in the study of classical network coding [5], and proved that routing is also optimal in this network. But it remains unknown if their arguments (or the arguments in Ref. [7]) can be applied to more complicated networks. On the contrary, Jain et al. [13] constructed a quantum network such that the achievable rate by network coding is k times the rate by routing, where k is the number of source-target pairs (which can be regarded as a quantum analogue of the gap between achievable rates with and without network coding in Ref. [5]).

IV. Quantum network coding with extra resources

This section reviews the results on quantum network coding in the settings where extra resources, which may be less expensive than quantum communication, are available. Such representative resources in quantum information processing are:

(i) classical communication; and
(ii) pre-shared entanglement.

A. What if free classical communication is available

In 2009, Kobayashi et al. [14] constructed the first protocol for general quantum networks in this setting. As a consequence, they showed that if a classical network is solvable by means of a *linear* (or even vector-linear) network coding protocol over a finite ring, then the corresponding quantum network is also quantumly solvable. (Using the terminology of achievable rates, their protocol implies that if $(r_1, \dots, r_k)$ is achievable in a classical network by means of a linear coding protocol, then the same rate is also achievable in the corresponding quantum network.) Moreover, the same authors [16] later removed the linearity condition for classical network coding, that is, they showed that if a classical network is solvable (by means of a possibly non-linear coding protocol), then the corresponding quantum network is also quantumly solvable.

[2] In what follows, a network is a multiple unicast one unless otherwise noted.

The two protocols in Refs. [14] and [16] basically first simulate a classical operation at each node in a coherent manner. For instance, if an intermediate node v receives z_1 and z_2 from two parent nodes and sends $f(z_1, z_2)$ to a child node in a classical network, the protocols first perform the quantum operation $|z_1\rangle|z_2\rangle|0\rangle \mapsto |z_1\rangle|z_2\rangle|f(z_1, z_2)\rangle$ at node v in the corresponding quantum network, where the first two quantum registers are sent from the parent nodes and the last quantum register is prepared at v. Then, the protocols send the last quantum register to the child node. The key point is that the first two quantum registers must be removed later because otherwise unnecessary entanglement remains. To remove these, the protocols measure these in the Fourier basis, and the phase error, which may occur by the measurement, is corrected later by sending the measurement result using free classical communication.

The main difference between the protocols in Refs. [14] and [16] is the timing when the measurement is done; the protocol of Ref. [14] does the measurement just after every time where the simulation is done at a node while the protocol of Ref. [16] delays any measurement until all the simulations are completed. This delay enables us to remove the phase errors by devising an order of measurements, even if f is a non-linear operation, and also reduce the amount of classical communication (at most double as many as the number of edges, which is linear in the network size).

As an application, based on the basic idea of the protocols in Refs. [14] and [16], we can construct communication-efficient protocols for sharing a cat state between a source node and multiple target nodes in a multicast network [15] or EPR pairs between each source-target pairs in a multiple unicast network [16]. Also, it may be noticed that the protocols in Refs. [14] and [16] use operations similar to measurement-based quantum computing [20]. Recently, de Beaudrap and Roetteler [2] made a connection between these protocols on a network and measurement-based quantum computation with graph states related to the network.

One might consider that free classical communication should follow the structure of the underlying network. The setting where free classical communication should be sent through the directed edges (called the forward-assisted case) was investigated in Ref. [17], which showed achievable rates in the butterfly and the inverted crown networks.

B. What if pre-shared entanglement is available

The following three cases has been considered in the literature:

(i) any two source nodes share prior entanglement;
(ii) any two neighboring nodes share prior entanglement; and
(iii) any two nodes share prior entanglement.

For case (i), Hayashi [7] considered the setting where each edge represents a quantum channel that can send one qubit or a classical channel that can send two bits (note that these are exchangeable under pre-shared EPR pairs via quantum teleportation and dense coding). In this setting, he exhibited a clear contrast between with and without shared EPR pairs between the two source nodes in the butterfly network: if the source nodes share EPR pairs, the network is quantumly solvable, while if not, the network is not quantumly solvable. Soeda et al. [23] followed up the study in this setting for considering more general quantum operations over two qubits (where sending two qubits from s_1 to t_1 and from s_2 to t_2 was regarded as the quantum operation swapping two qubits through the butterfly network).

For case (ii), there are also a few results on the butterfly network. First, it should be mentioned that the proof in Ref. [7] still works for this case, that is, the butterfly network is not quantumly solvable even when two neighboring nodes share any prior entanglement. Satoh et al. [21] gave a protocol motivated by quantum repeaters [3], which sends two qubits from the two source nodes to the corresponding target nodes in the butterfly network in the setting where each edge on the network represents a shared EPR pair, instead of a quantum channel with unit capacity, and free classical communication is available.

For case (iii), there have been no results except the following observation in Ref. [17]: if $(r_1, \ldots, r_k)$ is achievable in the classical network, then the same rate is also achievable in the corresponding quantum network. This can be easily seen since any source-target pair shares an EPR pair and also two bits become available between the pair by combining the assumed classical network coding and dense coding between neighboring nodes, which enables us to teleport a qubit from the source node to the target node.

V. Concluding Remarks

This paper has reported the current status of the studies on quantum network coding, which has been not yet satisfactory, in particular, for the general graphs beyond the butterfly network. Finally, we list the three specific open problems for which this paper reviewed the related results (these are stated by the terminology of quantum solvability but may be stated by the terminology of achievable rates).

1) Is the butterfly network quantumly solvable under the condition that the two source nodes share prior entanglement? (Note that the result in Ref. [7] mentioned in Section IV-B allows each edge to choose sending not only one qubit but two bits.)
2) If a quantum network is quantumly solvable under free classical communication, then is the corresponding classical network also solvable[3] (the converse of the main result in Ref. [16])?
3) If a quantum network is quantumly solvable under free pre-shared entanglement, then is the corresponding classical network also solvable (the converse of the observation in Ref. [17] made in Section IV-B)?

[3]In this case, we should note that the underlying graph becomes undirected since each directed edge that represents a quantum channel with unit capacity can be reversed by quantum teleportation with the help of free classical communication, as mentioned in Ref. [17].

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