# DECOMPOSING FUNCTIONS OF BAIRE CLASS 2 ON POLISH SPACES 

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#### Abstract

We prove the Decomposability Conjecture for functions of Baire class 2 on a Polish space to a separable metrizable space. This partially answers an important open problem in descriptive set theory.


## 1. Introduction

In descriptive set theory, the study of decomposability of Borel functions originated by a famous question asked by Luzin around a century ago: Is every Borel function decomposable into countably many continuous functions? This question was answered negatively. Many counterexamples appeared in the literature (cf. [8, 10]) show that, even a function of Baire class 1 is not necessarily decomposable. Among these counterexamples, the Pawlikowski function $P:(\omega+1)^{\omega} \rightarrow \omega^{\omega}$ stands in an important position. Indeed, Solecki [15] proved that:

Let $X, Y$ be separable metrizable spaces with $X$ analytic, and let $f: X \rightarrow Y$ be of Baire class 1 . Then $f$ is not decomposable into countably many continuous functions iff $P \sqsubseteq f$, i.e., there exists embeddings $\phi:(\omega+1)^{\omega} \rightarrow X$ and $\psi: \omega^{\omega} \rightarrow Y$ such that $\psi \circ P=f \circ \phi$.

Later, Pawlikowski and Sabok [13] generalized this theorem onto all Borel functions from an analytic space to a separable metrizable space. Motto Ros [11, Lemma 5.6] also gave an elegant proof for all functions of Baire class $n$ with $n<\omega$.

A natural generalization of Luzin's question is to replace continuous functions with $\boldsymbol{\Sigma}_{\gamma}^{0}$-measurable functions. We write $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}\right)$ if there exists a partition $\left(X_{k}\right)$ of $X$ with each $f \upharpoonright X_{k}$ is $\boldsymbol{\Sigma}_{\gamma}^{0}$-measurable; and also write $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta}^{0}\right)$ if such a partition can be a sequence of $\boldsymbol{\Delta}_{\delta}^{0}$ subsets of

[^0]$X$. It is trivial to see that, for $\delta \geq \gamma, f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta}^{0}\right)$ implies the $\boldsymbol{\Sigma}_{\delta^{-}}^{0}$ measurability of $f$. It is also well known that, for any $\boldsymbol{\Sigma}_{\delta}^{0}$-measurable function $f$ with $\delta>\gamma$, we have $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}\right) \Longleftrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta+1}^{0}\right)$ (cf. [11, Proposition 4.5]).

A slightly more finer notion of Baire hierarchy was essentially introduced by Jayne [3] for studying the Banach space of functions of Baire class $\alpha$. A function $f: X \rightarrow Y$ is called a $\boldsymbol{\Sigma}_{\alpha, \beta}$ function (or more precisely denoted by $\left.f^{-1} \boldsymbol{\Sigma}_{\beta}^{0} \subseteq \boldsymbol{\Sigma}_{\alpha}^{0}\right)$ if the preimage $f^{-1}(A)$ is $\boldsymbol{\Sigma}_{\alpha}^{0}$ in $X$ for every $\boldsymbol{\Sigma}_{\beta}^{0}$ subset $A$ of $Y$. The following theorem discovers a deep connection between this notion and decomposability:

Theorem 1.1 (Jayne-Rogers [4]). Let $X, Y$ be separable metrizable spaces with $X$ analytic, and let $f: X \rightarrow Y$. Then

$$
f^{-1} \boldsymbol{\Sigma}_{2}^{0} \subseteq \boldsymbol{\Sigma}_{2}^{0} \Longleftrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{2}^{0}\right)
$$

This theorem was generalized in [6] to the case that $X$ is an absolute Souslin- $\mathcal{F}$ set and $Y$ is an arbitrary regular topological space.

It is conjectured that the Jayne-Rogers Theorem can be extended to all finite Borel ranks as follows:

The Decomposability Conjecture (cf. [1, 11, 13]). Let $X, Y$ be separable metrizable spaces with $X$ analytic, and let $f: X \rightarrow Y$. Then for $n \geq 2$ we have

$$
f^{-1} \boldsymbol{\Sigma}_{n}^{0} \subseteq \boldsymbol{\Sigma}_{n}^{0} \Longleftrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{n}^{0}\right) .
$$

Furthermore, for $2 \leq m \leq n$ we have

$$
f^{-1} \boldsymbol{\Sigma}_{m}^{0} \subseteq \boldsymbol{\Sigma}_{n}^{0} \Longleftrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{n-m+1}^{0}, \boldsymbol{\Delta}_{n}^{0}\right) .
$$

This conjecture was further generalized to The Full Decomposability Conjecture (see [2, Section 4]) which covers all infinite Borel ranks. Motto Ros presented an equivalent condition of the decomposability conjecture (see [11, Conjecture 6.1]). Another interesting equivalent condition with some extra restrictions on spaces and on relation between $m, n$, concerning computability on Borel codes from $A$ to $f^{-1}(A)$, was given by Kihara in (9]. Most recently, Gregoriades-Kihara-Ng [2] proved

$$
f^{-1} \boldsymbol{\Sigma}_{m}^{0} \subseteq \boldsymbol{\Sigma}_{n}^{0} \Longrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{n-m+1}^{0}, \boldsymbol{\Delta}_{n+1}^{0}\right) .
$$

It is clear that the case $m=n=2$ in the decomposability conjecture is just the Jayne-Rogers Theorem. Remarkable progress is due to Semmes, the third author of this article. In his Ph.D. thesis [14], Semmes proved the case $m \leq n=3$ for functions $f: \omega^{\omega} \rightarrow \omega^{\omega}$. In his proof, many kinds of games for characterizing Borel functions were widely applied. From the viewpoint of Jayne's work [3] in functional analysis, the zero-dimensionality constraint on Semmes' theorem was strongly desired to be removed. In this article, we generalize Semmes' theorem to arbitrary Polish spaces:

Theorem 1.2. The decomposability conjecture is true for the case that $X$ is Polish space and $m \leq n=3$.

It is worth noting that in our proof, no game for Borel functions are involved. This is the key point that this proof can be extended to all Ploish spaces. This theorem consists of two cases: (a) $m=2, n=3$, and (b) $m=n=3$. We will prove them in sections 3 and 4 respectively.

Following the outline of Semmes' proof, the proof appearing in this article was developed by the first and the forth authors. Almost at the same time, the second author independently gave a detailed exposition of Semmes' strategy. He also asserted that the use of games for Borel functions is misleading, and emphasized the use of finite injury priority argument instead. Soon after reading it, Motto Ros pointed out that the same argument in the second author's proof also works well, with some minor modifications, for arbitrary Polish spaces.

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## 2. Preliminaries

All topological spaces considered in this article are separable metrizable. For any subset $A$ of a topological space $X$, we denote by $\bar{A}$ the closure of $A$ in $X$ and denote $A^{c}=X \backslash A$ for brevity.

We recall some basic notations. A topological space is called a Polish space if it is separable and completely metrizable, and is called an analytic space if it is homeomorphic to an analytic subset of a Polish space. Given a separable metrizable space $X$, Borel sets of $X$ can be analyzed into Borel hierarchy, consisting of $\boldsymbol{\Sigma}_{\xi}^{0}, \boldsymbol{\Pi}_{\xi}^{0}$ subsets for $1 \leq \xi<\omega_{1}$. As usual, we denote $\boldsymbol{\Delta}_{\xi}^{0}=\boldsymbol{\Sigma}_{\xi}^{0} \cap \boldsymbol{\Pi}_{\xi}^{0}$.

Let $X, Y$ be two separable metrizable spaces, and $f: X \rightarrow Y$. We say $f$ is $\boldsymbol{\Sigma}_{\alpha}^{0}$-measurable if $f^{-1}(U) \in \boldsymbol{\Sigma}_{\alpha}^{0}$ for every open set $U \subseteq Y$. For the definition of the Baire classes of functions, one can see [7, (24.1)]. It is well known that a function is of Baire class $\xi$ iff it is $\boldsymbol{\Sigma}_{\xi+1}^{0}$-measurable (cf. [7, (24.3)]).

In the section of introduction, we already presented notion of $\boldsymbol{\Sigma}_{\alpha, \beta}$ functions, $f^{-1} \boldsymbol{\Sigma}_{\beta}^{0} \subseteq \boldsymbol{\Sigma}_{\alpha}^{0}$ and $\operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta}^{0}\right)$. The following proposition give some well known properties which will be used again and again in the rest of this article.

Proposition 2.1 (folklore). Let $X, Y$ be two separable metrizable spaces, and let $f: X \rightarrow Y$. Then the following are equivalent:
(i) $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta}^{0}\right)$.
(ii) There exists a sequence $\left(A_{n}\right)$ of $\boldsymbol{\Sigma}_{\delta}^{0}$ subsets with $X=\bigcup_{n} A_{n}$ such that every $f \upharpoonright A_{n}$ is $\boldsymbol{\Sigma}_{\gamma}^{0}$-measurable.
(iii) There exists a sequence $\left(A_{n}\right)$ of $\boldsymbol{\Sigma}_{\delta}^{0}$ subsets with $X=\bigcup_{n} A_{n}$ such that every $f \upharpoonright A_{n} \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta}^{0}\right)$.

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial. We only prove (iii) $\Rightarrow$ (i).
For every $n<\omega$, since $A_{n} \in \boldsymbol{\Sigma}_{\delta}^{0}$, we can choose a sequence $\left(B_{n}^{m}\right)_{m<\omega}$ of $\Delta_{\delta}^{0}$ sets such that $A_{n}=\bigcup_{m} B_{n}^{m}$. Moreover, since

$$
f \upharpoonright A_{n} \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta}^{0}\right)
$$

there exist two sequences $\left(C_{n}^{k}\right)_{k<\omega},\left(D_{n}^{k}\right)_{k<\omega}$ of $\boldsymbol{\Sigma}_{\delta}^{0}$ sets with

$$
A_{n} \subseteq \bigcup_{k} C_{n}^{k}, \quad A_{n} \cap C_{n}^{k}=A_{n} \backslash D_{n}^{k}
$$

such that each $f \upharpoonright\left(C_{n}^{k} \cap A_{n}\right)$ is $\boldsymbol{\Sigma}_{\gamma}^{0}$-measurable. Note that $B_{n}^{m} \cap C_{n}^{k}=$ $B_{n}^{m} \backslash D_{n}^{k} \in \boldsymbol{\Delta}_{\delta}^{0}$, and $f \upharpoonright\left(B_{n}^{m} \cap C_{n}^{k}\right)$ is $\boldsymbol{\Sigma}_{\gamma}^{0}$-measurable for all $n, k, m<\omega$.

Let $\left(K_{l}\right)_{l<\omega}$ be an enumeration of all $B_{n}^{m} \cap C_{n}^{k}, n, k, m<\omega$. Then

$$
\bigcup_{l} K_{l}=\bigcup_{n, k, m}\left(B_{n}^{m} \cap C_{n}^{k}\right)=X
$$

For each $l<\omega$, put $K_{l}^{\prime}=K_{l} \backslash\left(\bigcup_{i<l} K_{i}\right)$. Then the sequence $\left(K_{l}^{\prime}\right)_{l<\omega}$ of $\boldsymbol{\Delta}_{\delta}^{0}$ subsets witnesses that $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{\gamma}^{0}, \boldsymbol{\Delta}_{\delta}^{0}\right)$.

## 3. The decomposability conjecture for $m=2, n=3$

We prove Theorem 1.2 for $m=2, n=3$ in this section, and for $m=n=3$ in the next section. The following lemma is the key tool for proving the main theorem of this section, just like the role of Lemma 4.3.3 in [14.

Lemma 3.1. Let $X, P$ be two separable metrizable spaces, and let $D \subseteq X$, $h: D \rightarrow P$ a function of Baire class 2. Let $\mathcal{B}_{P}$ be a countable topological basis of $P$, and for each $V \in \mathcal{B}_{P}$, let $\mathcal{G}_{V}$ be a countable class of subsets of $D$ such that

$$
h^{-1}(V)=\bigcup \mathcal{G}_{V}
$$

If $h \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, then there exist $V \in \mathcal{B}_{P}, G \in \mathcal{G}_{V}$, and a closed set $F \subseteq \bar{D}$ satisfying:
(a) For any open set $U$ with $F \cap U \neq \emptyset$,

$$
h \upharpoonright\left(h^{-1}\left(\bar{V}^{c}\right) \cap F \cap U\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \Delta_{3}^{0}\right) ;
$$

(b) $G \cap F$ is dense in $F$;
(c) $F \cap D \neq \emptyset$.

Proof. Let $\left\{U_{k}: k<\omega\right\}$ be a topological basis of $X$. For any $V \in \mathcal{B}_{P}$ and any closed subset $F \subseteq X$, we denote

$$
\begin{gathered}
\Gamma_{V}(F)=\left\{k<\omega: h \upharpoonright\left(h^{-1}\left(\bar{V}^{c}\right) \cap F \cap U_{k}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \Delta_{3}^{0}\right)\right\}, \\
\Theta_{V}(F)=\left\{x \in F: \forall k<\omega\left(x \in U_{k} \Rightarrow k \notin \Gamma_{V}(F)\right)\right\} .
\end{gathered}
$$

It is trivial to see that $\Theta_{V}(F) \subseteq \bar{D}$ is closed.

For any $G \in \mathcal{G}_{V}$, we define closed set $F_{V, G}^{\alpha}$ for $\alpha<\omega_{1}$ as follows:

$$
\begin{gathered}
F_{V, G}^{0}=X \\
F_{V, G}^{\alpha+1}=\overline{G \cap \Theta_{V}\left(F_{V, G}^{\alpha}\right)}, \\
F_{V, G}^{\lambda}=\bigcap_{\alpha<\lambda} F_{V, G}^{\alpha}, \quad \text { for limit ordinal } \lambda .
\end{gathered}
$$

Since $X$ is second countable, there exists a $\xi<\omega_{1}$ such that $F_{V, G}^{\alpha}=F_{V, G}^{\xi}$ for each $V, G$ and $\alpha \geq \xi$.

If there exist $V \in \mathcal{B}_{P}, G \in \mathcal{G}_{V}$ such that $F_{V, G}^{\xi} \neq \emptyset$, then $V, G$, and $F=F_{V, G}^{\xi}$ fulfil clauses (a) and (b). Set $U=X$ in (a), we can see (c) is also fulfilled.

Assume for contradiction that, for any $V \in \mathcal{B}_{P}, G \in \mathcal{G}_{V}$, we have $F_{V, G}^{\xi}=$ $\emptyset$. For $\alpha<\xi$ and $k \in \Gamma_{V}\left(F_{V, G}^{\alpha}\right)$, put

$$
H_{V, G, k}^{\alpha}=F_{V, G}^{\alpha} \cap U_{k}
$$

Note that

$$
h \upharpoonright\left(h^{-1}\left(\bar{V}^{c}\right) \cap H_{V, G, k}^{\alpha}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \Delta_{3}^{0}\right)
$$

Now define a subset $H$ of all $x$ satisfying that, there exist $V_{1}, V_{2} \in \mathcal{B}_{P}$ with $\overline{V_{1}} \cap \overline{V_{2}}=\emptyset$, and for $i=1,2$, there exist $G_{i} \in \mathcal{G}_{V_{i}}, \alpha_{i}<\xi$, and $k_{i} \in \Gamma_{V_{i}}\left(F_{V_{i}, G_{i}}^{\alpha_{i}}\right)$ such that $x \in H_{V_{1}, G_{1}, k_{1}}^{\alpha_{1}} \cap H_{V_{2}, G_{2}, k_{2}}^{\alpha_{2}}$. Since

$$
h \upharpoonright\left(h^{-1}\left({\overline{V_{i}}}^{c}\right) \cap H_{V_{1}, G_{1}, k_{1}}^{\alpha_{1}} \cap H_{V_{2}, G_{2}, k_{2}}^{\alpha_{2}}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \Delta_{3}^{0}\right) \quad(i=1,2)
$$

and $h^{-1}\left(\bar{V}_{i}^{c}\right)$ is $\Sigma_{3}^{0}$ in $D$ for $i=1,2$, by Proposition 2.1, we have

$$
h \upharpoonright\left(D \cap H_{V_{1}, G_{1}, k_{1}}^{\alpha_{1}} \cap H_{V_{2}, G_{2}, k_{2}}^{\alpha_{2}}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)
$$

Therefore, $h \upharpoonright(D \cap H) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$.
For any $x \in D, V \in \mathcal{B}_{P}$, and $G \in \mathcal{G}_{V}$ with $x \in G$, we claim that there exist $\alpha<\xi$ and $k \in \Gamma_{V}\left(F_{V, G}^{\alpha}\right)$ such that $x \in H_{V, G, k}^{\alpha}$. There is unique $\alpha<\xi$ such that $x \in\left(F_{V, G}^{\alpha} \backslash F_{V, G}^{\alpha+1}\right)$. Note that $x \notin(F \backslash \overline{G \cap F})$ for any closed set $F \subseteq X$, so $x \notin\left(\Theta_{V}\left(F_{V, G}^{\alpha}\right) \backslash F_{V, G}^{\alpha+1}\right)$. From the definition of $\Theta_{V}\left(F_{V, G}^{\alpha}\right)$, we can find a $k \in \Gamma_{V}\left(F_{V, G}^{\alpha}\right)$ such that $x \in U_{k}$. Then we have $x \in F_{V, G}^{\alpha} \cap U_{k}=H_{V, G, k}^{\alpha}$.

In the end, we consider $h \upharpoonright(D \backslash H)$. First, for any $x \in(D \backslash H)$, if $x \in H_{V, G, k}^{\alpha}$ for some $V \in \mathcal{B}_{P}, G \in \mathcal{G}_{V}, \alpha<\xi$, and $k \in \Gamma_{V}\left(F_{V, G}^{\alpha}\right)$, we claim that $h(x) \in \bar{V}$. If not, we can find a $V^{\prime} \in \mathcal{B}_{P}$ such that $h(x) \in V^{\prime}$ and $\bar{V} \cap \overline{V^{\prime}}=\emptyset$. Since $h^{-1}\left(V^{\prime}\right)=\bigcup \mathcal{G}_{V^{\prime}}$, we can find an $G^{\prime} \in \mathcal{G}_{V^{\prime}}$ such that $x \in G^{\prime}$. Hence $x \in H_{V^{\prime}, G^{\prime}, k^{\prime}}^{\alpha^{\prime}}$ for some $\alpha^{\prime}<\xi$ and $k^{\prime} \in \Gamma_{V^{\prime}}\left(F_{V^{\prime}, G^{\prime}}^{\alpha^{\prime}}\right)$, contradicting $x \notin H$. Secondly, let $d$ be a compatible metric on $P$. For any $n<\omega$, let

$$
\left(V_{m}^{n}, G_{m}^{n}, k_{m}^{n}, \alpha_{m}^{n}\right)_{m<\omega}
$$

be an enumeration of all $(V, G, k, \alpha)$ with $\operatorname{diam}(\bar{V}) \leq 1 / n, G \in \mathcal{G}_{V}, \alpha<\xi$, and $k \in \Gamma_{V}\left(F_{V, G}^{\alpha}\right)$. Denote

$$
H_{m}^{n}=H_{V_{m}^{n}, G_{m}^{n}, k_{m}^{n}}^{\alpha_{m}^{n}} .
$$

For any $x \in D$, we can find a $V \in \mathcal{B}_{P}$ with $\operatorname{diam}(\bar{V}) \leq 1 / n$ such that $h(x) \in V$ and a $G \in \mathcal{G}_{V}$ with $x \in G$. Hence $x \in H_{V, G, k}^{\alpha}$ for some $\alpha<\xi$ and $k \in \Gamma_{V}\left(F_{V, G}^{\alpha}\right)$. It follows that $D \subseteq \bigcup_{m} H_{m}^{n}$. Put $K_{m}^{n}=H_{m}^{n} \backslash \bigcup_{k<m} H_{k}^{n}$ for each $m$. Then $\left(K_{m}^{n}\right)_{m<\omega}$ is a sequence of pairwise disjoint $\boldsymbol{\Delta}_{2}^{0}$ sets. Fix a $y_{m}^{n} \in \overline{V_{m}^{n}}$ for each $m$. Define $g_{n}(x)=y_{m}^{n}$ for all $x \in K_{m}^{n}$. Then $g_{n}$ is of Baire class 1. Furthermore, we have $d\left(g_{n}(x), h(x)\right) \leq 1 / n$ for all $x \in(D \backslash H)$. So $\left(g_{n} \upharpoonright(D \backslash H)\right)_{n<\omega}$ uniformly converges to $h \upharpoonright(D \backslash H)$. It follows that $h \upharpoonright(D \backslash H)$ is of Baire class 1 also (see [7, (24.4) i)]).

Note that $H$ is an $F_{\sigma}$ set from its definition. So $D \backslash H$ is $G_{\delta}$ in $D$, and hence $h \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$. A contradiction!

In the rest of this section, we fix $X$ be a Polish space, $Y$ a separable metrizable space, and $f: X \rightarrow Y$ a $\boldsymbol{\Sigma}_{3}^{0}$-measurable function.

Definition 3.2. Let $\mathcal{F}=\left\langle F_{0}, \cdots, F_{k}\right\rangle$ be a finite sequence of closed sets of $X$ with $F_{0} \supseteq \cdots \supseteq F_{k}, U$ an open subset of $X$, and let $P \subseteq Y$.
(i) If $k=0$, i.e., $\mathcal{F}=\left\langle F_{0}\right\rangle$, then we say $\mathcal{F}$ is $P$-sharp in $U$ if $U \cap F_{0} \neq \emptyset$, and for any open set $U^{\prime} \subseteq U$ with $U^{\prime} \cap F_{0} \neq \emptyset$, we have

$$
f \upharpoonright\left(f^{-1}(P) \cap F_{0} \cap U^{\prime}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

We also say $F_{0}$ itself is $P$-sharp in $U$ for brevity.
(ii) If $k>0$, then we say $\mathcal{F}$ is $P$-sharp in $U$ if $F_{k}$ is $P$-sharp in $U$, and for any open set $U^{\prime} \subseteq U$ with $U^{\prime} \cap F_{k} \neq \emptyset, \mathcal{F} \upharpoonright k$ is $P$-sharp in some open set $U^{\prime \prime} \subseteq U^{\prime}$.

A similar notion named $\delta$ - $\sigma$-good was presented in [14]. The following propositions are trivial, we omit the proofs.

Proposition 3.3. Suppose $\mathcal{F}=\left\langle F_{0}, \cdots, F_{k}\right\rangle$ is $P$-sharp in $U$. Then for any open set $U^{\prime} \subseteq U$ with $U^{\prime} \cap F_{k} \neq \emptyset$, we have $\mathcal{F}$ is $P$-sharp in $U^{\prime}$.

Proposition 3.4. Suppose $\mathcal{F}$ is $P$-sharp in $U$. Then for any $m<\operatorname{lh}(\mathcal{F})$, $\mathcal{F} \upharpoonright m$ is $P$-sharp in some open set $U^{\prime} \subseteq U$.

The following lemma is modified from [14, Lemma 4.3.6].
Lemma 3.5. Suppose $\mathcal{F}=\left\langle F_{0}, \cdots, F_{k}\right\rangle$ is P-sharp in $U$. Let $\left(C_{l}\right)_{l<m}$ be a sequence of pairwise disjoint closed subsets of $P$. Then there exist at most $k+1$ many $l$ such that $\mathcal{F}$ is not $P \backslash C_{l}$-sharp in any open set $U^{\prime} \subseteq U$.

Proof. We begin with $k=0$. Without loss of generality, suppose there exists an $l<m$, say, $l=0$, such that $F_{0}$ is not $P \backslash C_{0}$-sharp in $U$. Then there exists an open set $U_{0} \subseteq U$ with $U_{0} \cap F_{0} \neq \emptyset$ such that

$$
f \upharpoonright\left(f^{-1}\left(P \backslash C_{0}\right) \cap F_{0} \cap U_{0}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \Delta_{3}^{0}\right) .
$$

Assume for contradiction that there exists $l \neq 0$ such that $F_{0}$ is not $P \backslash C_{l^{-}}$ sharp in $U_{0}$, then there is an open set $U_{l} \subseteq U_{0}$ with $U_{l} \cap F_{0} \neq \emptyset$ such that

$$
f \upharpoonright\left(f^{-1}\left(P \backslash C_{l}\right) \cap F_{0} \cap U_{l}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \Delta_{3}^{0}\right) .
$$

Since $C_{0}$ and $C_{l}$ are disjoint closed subsets of $P$, Proposition 2.1 gives

$$
f \upharpoonright\left(f^{-1}(P) \cap F_{0} \cap U_{l}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \Delta_{3}^{0}\right),
$$

contradicting that $F_{0}$ is $P$-sharp in $U$.
For $k>0$, assume that we have proved for all $k^{\prime}<k$. Since $F_{k}$ is $P$-sharp in $U$, from the arguments for $k=0$ above, we may assume that there is an open set $U_{0} \subseteq U$ with $U_{0} \cap F_{k} \neq \emptyset$ such that $F_{k}$ is $P \backslash C_{l}$-sharp in $U_{0}$ for any $l \neq 0$. Assume for contradiction that there are $k+1$ many $l \neq 0$, say, $l=1, \cdots, k, k+1$, such that $\mathcal{F}$ is not $P \backslash C_{l}$-sharp in any open set $U^{\prime} \subseteq U_{0}$. Particularly, $\mathcal{F}$ is not $P \backslash C_{1}$-sharp in $U_{0}$, so there exists an open set $U_{1} \subseteq U_{0}$ with $U_{1} \cap F_{k} \neq \emptyset$ such that $\mathcal{F} \upharpoonright k$ is not $P \backslash C_{1}$-sharp in any open set $U^{\prime} \subseteq U_{1}$. Similarly, we can find a sequence of open sets $U_{k+1} \subseteq U_{k} \subseteq \cdots \subseteq U_{1} \subseteq U_{0}$ such that $U_{l} \cap F_{k} \neq \emptyset$ and $\mathcal{F} \upharpoonright k$ is not $P \backslash C_{l}$-sharp in any $U^{\prime} \subseteq U_{l}$ for $0<l \leq k+1$. By Definition of $P$-sharp, there is a open set $U^{*} \subseteq U_{k+1}$ such that $\mathcal{F} \upharpoonright k$ is $P$-sharp in $U^{*}$, contradicting the induction hypothesis.

Let $\ulcorner\cdot, \cdot\urcorner$ be the bijection: $\omega \times \omega \rightarrow \omega$ as following:

$$
\begin{gathered}
\ulcorner 0,0\urcorner=0, \\
\ulcorner 0, j+1\urcorner=\ulcorner j, 0\urcorner+1, \\
\ulcorner i+1, j-1\urcorner=\ulcorner i, j\urcorner+1 .
\end{gathered}
$$

Denote

$$
\Omega=\left\{z \in 2^{\omega}: \exists i \exists^{\infty} j(z(\ulcorner i, j\urcorner)=1)\right\} .
$$

It is well known that $\Omega$ is $\boldsymbol{\Sigma}_{3}^{0}$-complete subset of $2^{\omega}$.
For any $z \in 2^{\omega}$ and $l<\omega$, we call sequence

$$
z \upharpoonright(\ulcorner 0, l\urcorner+1), z \upharpoonright(\ulcorner 1, l-1\urcorner+1), \cdots, z \upharpoonright(\ulcorner l, 0\urcorner+1)
$$

the $l$-th diagonal of $z$, and call $z \upharpoonright(\ulcorner l, 0\urcorner+1)$ the end of $l$-th diagonal. For $s \in 2^{<\omega}$, we denote $\operatorname{lh}(s)=i$ the length of $s$. If $s \subseteq z$ and $\operatorname{lh}(s)=\ulcorner i, j\urcorner+1$, then $s$ is in $(i+j)$-th diagonal. Moreover, the $l$-th diagonal of $z$ is also named the $l$-th diagonal of $s$ when $\ulcorner l, 0\urcorner<\operatorname{lh}(s)$.

For $s \neq \emptyset$, let $\operatorname{lh}(s)=\ulcorner i, j\urcorner+1$. We denote $\operatorname{row}(s)=i, \operatorname{col}(s)=j$. If $i+j>0$, we call $s \upharpoonright(\ulcorner i+j-1,0\urcorner+1)$ the end of the last diagonal of $s$, denoted by $u(s)$. If $j>0$, we call $s \upharpoonright(\ulcorner i, j-1\urcorner+1)$ the left neighbor of $s$, denoted by $v(s)$.

For proving the following theorem, we need an order $\preceq$ on $2^{<\omega}$ define by

$$
t \preceq s \Longleftrightarrow \operatorname{lh}(t)<\operatorname{lh}(s) \text { or }\left(\operatorname{lh}(t)=\operatorname{lh}(s), t \leq_{\operatorname{lex}} s\right),
$$

where $\leq_{\text {lex }}$ is the usual lexicographical order. We also denote $t \prec s$ when $t \preceq s$ but not $t=s$. Denote

$$
s_{M}=\max _{\preceq}\left\{t: t \prec s^{`} 0\right\} .
$$

Theorem 3.6. Let $X$ be a Polish space, $Y$ a separable metrizable space, and let $f: X \rightarrow Y$. Then

$$
f^{-1} \boldsymbol{\Sigma}_{2}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0} \Longleftrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

Proof. The " $\Leftarrow$ " part is trivial, we only prove the " $\Rightarrow$ " part.
Assume for contradiction that $f \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$. We will define a continuous embedding $\psi: 2^{\omega} \rightarrow X$ and an open set $O \subseteq Y$ such that

$$
\psi^{-1}\left(f^{-1}(O)\right)=\Omega
$$

Thus $f^{-1}(Y \backslash O)$ is $\boldsymbol{\Pi}_{3}^{0}$-complete subset of $X$, contradicting $f^{-1} \boldsymbol{\Sigma}_{2}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0}$.
For any open set $V \subseteq Y$, since $f^{-1}(V)$ is $\boldsymbol{\Sigma}_{3}^{0}$, we can fix a system of open set $W_{n}^{m}(V) \subseteq X(m, n<\omega)$ with

$$
f^{-1}(V)=\bigcup_{m} \bigcap_{n} W_{n}^{m}(V) .
$$

Denote $G^{m}(V)=\bigcap_{n} W_{n}^{m}(V)$. For $G=G^{m}(V)$, we also denote $W_{n}(G)=$ $W_{n}^{m}(V)$.

For $s \in 2^{<\omega}$ with $s \neq \emptyset$, let $\operatorname{lh}(s)=\ulcorner i, j\urcorner+1$. We say $s$ is an inheritor if $j>0$ and $s(\ulcorner k, i+j-k\urcorner)=0$ for any $k<i$, otherwise we say $s$ is an innovator. Note that $s$ is always an inheritor if $i=0, j>0$, and is always an innovator if $j=0$.

Fix a compatible metric $d$ on $X$ with $d \leq 1$. We will inductively construct, for each $s \in 2^{<\omega}$, an open set $V_{s}$ of $Y$, a $G_{\delta}$ set $G_{s}$ of $X$, a closed set $F_{s}$ of $X$, an open set $U_{s}$ of $X$, and a sequence of open sets $\left(U_{s}^{w}\right)_{s \preceq w \prec s\urcorner 0}$ of $X$ satisfying the following:
(0) $\operatorname{diam}\left(\overline{U_{s}}\right) \leq 2^{-\operatorname{lh}(s)}, U_{s\urcorner 0} \cap U_{s\urcorner 1}=\emptyset, \overline{U_{s\urcorner 0}} \subseteq U_{s}^{w}, \overline{U_{s\urcorner 1}} \subseteq U_{s}^{w}$;
(1) $V_{s\urcorner 0}=V_{s\urcorner 1}, G_{s\urcorner 0}=G_{s\urcorner 1}, F_{s\urcorner 0}=F_{s\urcorner 1}$;
(2) for $s, t \in 2^{<\omega}$, we have $V_{s}=V_{t}$ or $\overline{V_{s}} \cap \overline{V_{t}}=\emptyset$;
(3) there exist $m<\omega$ such that $G_{s}=G^{m}\left(V_{s}\right)$;
(4) if $\operatorname{col}(s)>0$, then $F_{s\urcorner 0}=F_{s \wedge 1} \subseteq F_{s}$;
(5) $F_{s} \cap U_{s}^{w} \neq \emptyset$ for each $w$;
(6) $U_{s}=U_{s}^{s}$, and $U_{s}^{w_{1}} \supseteq U_{s}^{w_{2}}$ for $w_{1} \preceq w_{2}$;
(7) $G_{s} \cap F_{s}$ is dense in $F_{s}$;
(8) if $s$ is an inheritor, then we have

$$
V_{s}=V_{v(s)}, \quad G_{s}=G_{v(s)}, \quad F_{s}=F_{v(s)}
$$

furthermore, (a) if $s^{\wedge} 0$ is also an inheritor, then $U_{s}^{w} \cap F_{u(s)} \neq \emptyset$ for each $w$; (b) if $s^{\wedge} 0$ is an innovator, then $U_{s} \subseteq W_{n}\left(G_{s}\right)$ for all $n<\operatorname{lh}(s)$;
(9) if $s$ is an innovator, then $U_{s} \cap F_{s} \cap G^{m}\left(V_{t}\right)=\emptyset$ for all $m<\operatorname{lh}(s)$ and all $\operatorname{lh}(t)<\operatorname{lh}(s)$;
(10) by letting $P_{s}=Y \backslash \bigcup_{t \preceq s} \overline{V_{t}}$,

$$
\mathcal{F}_{s}=\left\langle F_{s\lceil(\operatorname{lh}(s)-\operatorname{row}(s))}, \cdots, F_{s\lceil(\operatorname{lh}(s)-1)}, F_{s}\right\rangle,
$$

for $t \preceq s \prec t^{\curvearrowright} 0$, we have
(a) if $t^{\wedge} 0$ is an innovator, then $\mathcal{F}_{t}$ is $P_{s}$-sharp in $U_{t}^{s}$;
(b) if $t^{\wedge} 0$ is an inheritor, then $\mathcal{F}_{u\left(t^{\wedge}\right)}$ is $P_{s}$-sharp in $U_{t}^{s}$.

When we complete the construction, for any $z \in 2^{\omega}$, we set $\psi(z)$ to be the unique element of $\bigcap_{k} U_{z\lceil k}$. From (0) and (6), we see that $\psi$ is a continuous embedding from $2^{\omega}$ to $X$. Put

$$
O=\bigcup_{t \in 2^{<\omega}} V_{t} .
$$

If $z \in \Omega$, let $i_{0}$ be the least $i$ such that there are infinitely many $j$ with $z(\ulcorner i, j\urcorner)=1$. Then there is $J_{0}<\omega$ such that $z(\ulcorner i, j\urcorner)=0$ for all $i<i_{0}$ and all $j>J_{0}$. Hence for any $j>J_{0}, z \upharpoonright\left(\left\ulcorner i_{0}, j\right\urcorner+1\right)$ is an inheritor. Denote

$$
V=V_{z \backslash\left(\left\ulcorner i_{0}, J_{0}\right\urcorner+1\right)}, \quad G=G_{z \backslash\left(\left\ulcorner i_{0}, J_{0}\right\urcorner+1\right)} .
$$

By (8), we have $V=V_{z \backslash\left(\left\ulcorner i_{0}, j\right\urcorner+1\right)}$ and $G=G_{z \backslash\left(\left(i_{0}, j\right\urcorner+1\right)}$ for all $j \geq J_{0}$. If $j>J_{0}$ with $z\left(\left\ulcorner i_{0}, j\right\urcorner\right)=1$, by (8)(b), we have $\psi(z) \in W_{n}(G)$ for all $n \leq$ $\left\ulcorner i_{0}, j\right\urcorner$. Since there is infinitely many such $j$, we have $\psi(z) \in G \subseteq f^{-1}(V)$. Therefore, $f(\psi(z)) \in V \subseteq O$.

If $z \notin \Omega$, we show that $f(\psi(z)) \notin O$. If not, there exits $t_{1}$ and $m_{1}$ such that $\psi(z) \in G^{m_{1}}\left(V_{t_{1}}\right)$. Fix an $i_{1}>\max \left\{m_{1}, \operatorname{lh}\left(t_{1}\right)\right\}$. Since $z \notin \Omega$, for large enough $j$, we have $z(\ulcorner i, j\urcorner)=0$ for all $i<i_{1}$, so $z \upharpoonright\left(\left\ulcorner i_{1}, j\right\urcorner+1\right)$ is an inheritor. Let $J_{1}$ be the largest $j$ such that $z \upharpoonright\left(\left\ulcorner i_{1}, j\right\urcorner+1\right)$ is an innovator. Denote

$$
F=F_{z \mid\left(\left\ulcorner i_{1}, J_{1}\right\urcorner+1\right)} .
$$

By (8), we have $F=F_{z \backslash\left(\left\ulcorner i_{1}, j\right\urcorner+1\right)}$ for all $j \geq J_{1}$. From (5) and (6), we see that $F \cap U_{\left.z \backslash\left(\left(i_{1}, j\right\rceil+1\right)\right)} \neq \emptyset$ for any $j>J_{1}$, and hence $\psi(z) \in F$. It follows from (9) that $F \cap U_{z \backslash\left(\left\ulcorner i_{1}, J_{1}\right\urcorner+1\right)} \cap G^{m_{1}}\left(V_{t_{1}}\right)=\emptyset$. Thus $\psi(z) \notin G^{m_{1}}\left(V_{t_{1}}\right)$. A contradiction!

Now we turn to the construction.
First, set $D, P, h, \mathcal{B}_{P}, \mathcal{G}_{V}$ as follows:
(i) $P=Y, D=X$, and $h=f$;
(ii) $\mathcal{B}_{P}$ is a countable basis of $Y$;
(iii) for each $V \in \mathcal{B}_{P}$, let $\mathcal{G}_{V}=\left\{G^{m}(V): m<\omega\right\}$.

Applying Lemma 3.1 with these $D, P, h, \mathcal{B}_{P}, \mathcal{G}_{V}$, we get an open set $V \subseteq Y$, a $G_{\delta}$ set $G \in \mathcal{G}_{V}$, and a non-empty closed set $F \subseteq X$. Then put

$$
V_{\emptyset}=V, \quad G_{\emptyset}=G, \quad F_{\emptyset}=F, \quad U_{\emptyset}=X .
$$

Secondly, assume that we have constructed $V_{t}, G_{t}, F_{t}, U_{t}$, and $U_{t}^{w}$ for $t, w \prec s^{\wedge} 0$. We will define for $s^{\wedge} 0$ and $s^{\wedge} 1$. We consider the following two cases:

Case 1. Assume $s^{\wedge} 0$ is an inheritor. Let $v=v\left(s^{\wedge} i\right), u=u\left(s^{\wedge} i\right)$. For $i=0,1$, put

$$
V_{s \neg i}=V_{v}, \quad G_{s\urcorner i}=G_{v}, \quad F_{s \neg i}=F_{v} .
$$

Note that $s=u$ or $s$ is also an inheritor with $u(s)=u$. By (5) and (8)(a), $U_{s}^{s_{M}} \cap F_{u} \neq \emptyset$.

Subcase 1.1. If $\operatorname{col}\left(s^{\wedge} 0^{\wedge} 0\right)>0$, then $s^{\wedge} 0^{\wedge} 0$ is still an inheritor. We can find a $U_{s\urcorner 0}$ such that

$$
\overline{U_{s\urcorner 0}} \subseteq U_{s}^{s_{M}}, \quad U_{s\urcorner 0} \cap F_{u} \neq \emptyset, \quad \operatorname{diam}\left(\overline{U_{s \sim 0}}\right) \leq 2^{-(\operatorname{lh}(s)+1)} .
$$

By (10)(b), we see $\mathcal{F}_{u}$ is $P_{s_{M}}$-sharp in $U_{s}^{s_{M}}$. By Proposition 3.4, $\mathcal{F}_{v}$ is $P_{s_{M}}$-sharp in some open set $U \subseteq U_{s}^{s_{M}}$, and hence $U \cap F_{v} \neq \emptyset$. Denote $W=\bigcap_{n \leq \operatorname{lh}(s)} W_{n}\left(G_{v}\right)$. Since $W \supseteq G_{v}$, by (7) we have $W \cap F_{v}$ is open dense in $F_{v}$, and hence $W \cap U \cap F_{v} \neq \emptyset$. We can find $U_{s \sim 1}$ such that

$$
\overline{U_{s \sim 1}} \subseteq U \cap W, \quad U_{s \wedge 1} \cap F_{v} \neq \emptyset, \quad \operatorname{diam}\left(\overline{U_{s^{\wedge}-1}}\right) \leq 2^{-(\operatorname{lh}(s)+1)} .
$$

By shrinking we may assume that $U_{s\urcorner 0} \cap U_{s\urcorner 1}=\emptyset$. For $i=0,1$, we set $U_{s\urcorner 0}^{s^{\wedge} i}=U_{s\urcorner 0}, U_{s\urcorner 1}^{s\urcorner 1}=U_{s \wedge 1}$, and for other $t$, set $U_{t}^{s \smile i}=U_{t}^{s_{M}}$.

Subcase 1.2. If $\operatorname{col}\left(s^{\wedge} 0^{\wedge} 0\right)=0$, then $s^{\wedge} 0^{\wedge} 0$ is an innovator. We define $U_{s \neg i}$ for $i=0,1$ similar to $U_{s \neg 1}$ in Subcase 1.1 with one more requirement $U_{s \neg 0} \cap U_{s \frown 1}=\emptyset$.

It is trivial to check clauses (0)-(9). Note that $P_{s^{\wedge} 0}=P_{s^{\wedge}}=P_{s_{M}}$ and $\mathcal{F}_{s^{\wedge} 0}=\mathcal{F}_{s \wedge 1}=\mathcal{F}_{v}$. Since (10) holds for $s_{M}$, it also holds for $s^{\wedge} 0$ and $s^{\wedge} 1$.

Case 2 . Assume $s^{\wedge} 0$ is an innovator. We inductively define $V^{l}, G^{l}, F^{l}$ and $U^{l}$ for each $l<\omega$ as the following:

Since $s \preceq s_{M} \prec s^{\wedge} 0$, by (10)(a), we have $\mathcal{F}_{s}$ is $P_{s_{M}}$-sharp in $U_{s}^{s_{M}}$. So

$$
f \upharpoonright\left(f^{-1}\left(P_{s_{M}}\right) \cap F_{s} \cap U_{s}^{s_{M}}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

Denote $F^{-1}=F_{s}, U^{-1}=U_{s}^{s_{M}}$. Assume that we have defined $V^{k}, G^{k}, F^{k}$ and $p^{k}$ for $k<l$. Set $D, P, h, \mathcal{B}_{P}, \mathcal{G}_{V}$ as follows:
(i) $P=P_{s_{M}} \backslash \bigcup_{k<l} \overline{V^{k}}, D=f^{-1}(P) \cap F^{l-1} \cap U^{l-1}$, and $h=f \upharpoonright D$;
(ii) $\mathcal{B}_{P}$ is a countable basis of $P$ such that $\bar{V} \subseteq P$ for each $V \in \mathcal{B}_{P}$;
(iii) for each $V \in \mathcal{B}_{P}$, let $\mathcal{G}_{V}=\left\{D \cap G^{m}(V): m<\omega\right\}$.

Applying Lemma 3.1 with these $D, P, h, \mathcal{B}_{P}, \mathcal{G}_{V}$, we get an open set $V \subseteq Y$, a $G_{\delta}$ set $G=G^{m}(V)$ for some $m<\omega$, and a closed set $F \subseteq \bar{D} \subseteq F^{l-1} \subseteq F_{s}$ with $F \cap U^{l-1} \supseteq F \cap D \neq \emptyset$. Denote $V^{l}=V, G^{l}=G, F^{l}=F$. If $\mathcal{F}_{s}^{\curvearrowright} F^{l}$ is $P_{s_{M}} \backslash \bar{V}^{l}$-sharp in $U^{l-1}$, set $U^{l}=U^{l-1}$. Otherwise, since Lemma 3.1 implies that $F^{l}$ is $P_{s_{M}} \backslash \overline{V^{l}}$-sharp in $U^{l-1}$, we can find an open set $U^{l} \subseteq U^{l-1}$ with $U^{l} \cap F^{l} \neq \emptyset$ such that $\mathcal{F}_{s}$ is not $P_{s_{M}} \backslash \bar{V}^{l}$-sharp in any open set $U \subseteq U^{l}$. This complete the induction.

For $s \prec t \preceq s_{M}$, if $t^{\wedge} 0$ is an innovator, it follows from (10)(a) that $\mathcal{F}_{t}$ is $P_{s_{M}}$-sharp in $U_{t}^{s_{M}}$. By Lemma 3.5, we can find a natural number $L_{t}$ such that, for any $l \geq L_{t}, \mathcal{F}_{t}$ is $P_{s_{M}} \backslash \bar{V}^{l}$-sharp in some open set $U_{t}^{l} \subseteq U_{t}^{s_{M}}$. If $t^{\curvearrowright} 0$ is an inheritor, from (10)(b) and Lemma 3.5, we can also find a natural number $L_{t}$ such that, for any $l \geq L_{t}, \mathcal{F}_{u\left(t^{\prime}\right) 0}$ is $P_{s_{M}} \backslash \bar{V}^{l}$-sharp in some open set $U_{t}^{l} \subseteq U_{t}^{s_{M}}$. Moreover, assume for contradiction that there exist $l_{0}<\cdots<l_{m}$ with $m=\operatorname{lh}\left(\mathcal{F}_{s}\right)$ such that $\mathcal{F}_{s}$ is not $P_{s_{M}} \backslash \overline{V^{l_{j}}}$-sharp in any
open set $U \subseteq U^{l_{j}}$ for $j \leq m$. This contradicts Lemma 3.5, because $U^{l_{m}} \subseteq$ $U_{s}^{s_{M}}$ and $U^{l_{m}} \cap F^{l_{m}} \neq \emptyset$ implies that $\mathcal{F}_{s}$ is $P_{s_{M}}$-sharp in $U^{l_{m}}$. Therefore, comparing with the definition of $U^{l}$, we can find an natural number $L^{\prime}$ such


$$
L=\max \left\{L^{\prime}, L_{t}: s \prec t \preceq s_{M}\right\}
$$

and $U_{t}^{s ` 0}=U_{t}^{s ` 1}=U_{t}^{L}$ for $\left.t \prec s\right\urcorner 0 \prec t \curvearrowright 0$, i.e., for $s \prec t \preceq s_{M}$.
In the end, denote

$$
A=\bigcup_{\operatorname{lh}(t) \leq \operatorname{lh}(s)} \bigcup_{m \leq \operatorname{lh}(s)} G^{m}\left(V_{t}\right) .
$$

Then $A$ is $G_{\delta}$ set. By (3) and $V^{L} \subseteq P_{s_{M}}$, we have $G^{L} \cap A=\emptyset$. It follows from Lemma 3.1 that $G^{L} \cap F^{L}$ is dense in $F^{L}$, so $A \cap F^{L}$ is nowhere dense in $F^{L}$. We can find two open sets $U_{s\urcorner 0}$ and $U_{s\urcorner 1}$ such that $U_{s\urcorner 0} \cap U_{s\urcorner 1}=\emptyset$, and for $i=0,1$, we have
$\overline{U_{s ` i}} \subseteq U^{L}, \quad U_{s^{\wedge} i} \cap F^{L} \neq \emptyset, \quad U_{s^{\wedge} i} \cap F^{L} \cap A=\emptyset, \quad \operatorname{diam}\left(\overline{U_{s^{\wedge}}}\right) \leq 2^{-(\mathrm{lh}(s)+1)}$.
Now put

$$
V_{s\urcorner i}=V^{L}, \quad G_{s\urcorner i}=G^{L}, \quad F_{s\urcorner i}=F^{L},
$$

and $U_{s\urcorner 0}^{s \frown i}=U_{s\urcorner 0}, U_{s\urcorner 1}^{s \frown 1}=U_{s\urcorner 1}$.
Corollary 3.7. Let $X$ be a Polish space, $Y$ a separable metrizable space, and let $f: X \rightarrow Y$. If $f \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, then there exists a Cantor set $C \subseteq X$ such that $f \upharpoonright C \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$.
Proof. Let $\psi$ be the continuous embedding defined in Theorem 3.6. Put $C=\psi\left(2^{\omega}\right)$.

## 4. The decomposability conjecture for $m=n=3$

Before proving Theorem 1.2 for $m=n=3$, we prove a known result first: for functions of Baire class 1,

$$
f^{-1} \boldsymbol{\Sigma}_{3}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0} \Rightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

This is an easy corollary of Solecki's theorem (see [15, Theorem 4.1]), since $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) \Longleftrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}\right)$ and $P^{-1} \boldsymbol{\Sigma}_{3}^{0} \nsubseteq \boldsymbol{\Sigma}_{3}^{0}$. Furthermore, this result is also a special case of [13, Corollary 1.2], [11, Corollary 5.11], or [2, Theorem 1.1]. In order to show a completely different method of proof, we present a direct proof which follows the same idea as in the previous section. The readers can skip directly to Theorem 4.7.
Lemma 4.1. Let $X, P$ be two separable metrizable spaces, and let $D \subseteq X$, and $h: D \rightarrow P$ a function of Baire class 1 . Let $\mathcal{B}_{P}$ be a countable topological basis of $P$. If $h \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, then there exist $a V \in \mathcal{B}_{P}$ and two closed sets $E \subseteq F \subseteq \bar{D}$ satisfying:
(a) for any open set $U$ with $E \cap U \neq \emptyset$, we have

$$
h \upharpoonright\left(h^{-1}(V) \cap U \cap E\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \Delta_{3}^{0}\right) ;
$$

(b) for any open set $U$ with $F \cap U \neq \emptyset$, we have

$$
h \upharpoonright\left(h^{-1}\left(\bar{V}^{c}\right) \cap F \cap U\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)
$$

(c) $E \cap D \neq \emptyset$.

Proof. Let $\left\{U_{k}: k<\omega\right\}$ be a topological basis of $X$. For any $B \in \mathcal{B}_{P}$, we denote

$$
F^{B}=\left\{x \in X: \forall k\left(x \in U_{k} \Rightarrow h \upharpoonright\left(h^{-1}\left(B^{c}\right) \cap U_{k}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)\right)\right\}
$$

It is trivial to see that
(i) $F^{B}$ is closed,
(ii) $h \upharpoonright\left(h^{-1}\left(B^{c}\right) \backslash F^{B}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, and
(iii) for any open set $U$ with $F^{B} \cap U \neq \emptyset$,

$$
h \upharpoonright\left(h^{-1}\left(B^{c}\right) \cap F^{B} \cap U\right) \notin \operatorname{dec}\left(\Sigma_{1}^{0}, \Delta_{3}^{0}\right) .
$$

Assume for contradiction that, $h \upharpoonright\left(h^{-1}(B) \cap F^{B}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$ for any $B \in \mathcal{B}_{P}$. We denote

$$
\begin{aligned}
& H_{1}=\bigcup_{B \in \mathcal{B}_{P}}\left(h^{-1}(B) \cap F^{B}\right) \\
& H_{2}=\bigcup_{B \in \mathcal{B}_{P}}\left(h^{-1}\left(B^{c}\right) \backslash F^{B}\right)
\end{aligned}
$$

It is straightforward to check that, $h \upharpoonright H_{i} \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$ for $i=1,2$.
Denote $H_{3}=D \backslash\left(H_{1} \cup H_{2}\right)$. For any $x \in H_{3}$ and any $B \in \mathcal{B}_{P}$, we have

$$
\begin{aligned}
& h(x) \in B \Rightarrow x \in h^{-1}(B) \Rightarrow x \notin F^{B} \\
& h(x) \notin B \Rightarrow x \in h^{-1}\left(B^{c}\right) \Rightarrow x \in F^{B}
\end{aligned}
$$

So $h \upharpoonright H_{3}$ is continuous.
Let $\tilde{Y} \supseteq Y$ be a Polish space. By Kuratowski's theorem (cf. [7, (3.8)]), there is a $G_{\delta}$ set $G \supseteq H_{3}$ and a continuous function $g: G \rightarrow \tilde{Y}$ such that $g \upharpoonright H_{3}=h \upharpoonright H_{3}$. Put $H=\{x \in D \cap G: h(x)=g(x)\}$. Since $h$ is of Baire class 1, we see $H$ is $G_{\delta}$ subset of $D$ and $H_{1} \cup H_{2} \cup H=D$. Note that $H_{1}$ is $F_{\sigma}$ subset of $D$ and $H_{2}$ is $\boldsymbol{\Sigma}_{3}^{0}$ subset of $D$. It follows that $h \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$. A contradiction!

Therefore, there exists a $B \in \mathcal{B}_{P}$ such that

$$
h \upharpoonright\left(h^{-1}(B) \cap F^{B}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

Since $B=\bigcup\left\{V \in \mathcal{B}_{P}: \bar{V} \subseteq B\right\}$, we can find a $V \in \mathcal{B}_{P}$ with $\bar{V} \subseteq B$ such that

$$
h \upharpoonright\left(h^{-1}(V) \cap F^{B}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)
$$

In the end, define

$$
E=\left\{x \in F^{B}: \forall k\left(x \in U_{k} \Rightarrow h \upharpoonright\left(h^{-1}(V) \cap U_{k}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)\right)\right\}
$$

Then we have $E \cap D \neq \emptyset$, and

$$
h \upharpoonright\left(h^{-1}(V) \cap U \cap E\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)
$$

for any open set with $U \cap E \neq \emptyset$. Note that $\bar{V}^{c} \supseteq B^{c}$. So $V, E$ and $F^{B}$ satisfy clauses (a)-(c) as desired.

In the rest of this section, we fix $X$ be a Polish space, $Y$ a separable metrizable space, and $f: X \rightarrow Y$ a $\Sigma_{3}^{0}$-measurable function.

Definition 4.2. Let $\mathcal{F}=\left\langle F_{0}, \cdots, F_{k}\right\rangle$ be a finite sequence of closed sets of $X$ with $F_{0} \supseteq \cdots \supseteq F_{k}, U$ an open subset of $X$, and let $\mathcal{P}=\left\langle P_{0}, \cdots, P_{k}\right\rangle$ be a sequence of pairwise disjoint subsets of $Y$.
(i) If $k=0$, i.e., $\mathcal{F}=\left\langle F_{0}\right\rangle, \mathcal{P}=\left\langle P_{0}\right\rangle$, then we say $\mathcal{F}$ is $\mathcal{P}$-sharp in $U$ if $U \cap F_{0} \neq \emptyset$, and for any open set $U^{\prime} \subseteq U$ with $U^{\prime} \cap F_{0} \neq \emptyset$, we have

$$
f \upharpoonright\left(f^{-1}\left(P_{0}\right) \cap F_{0} \cap U^{\prime}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

We also say $F_{0}$ itself is $P_{0}$-sharp in $U$ for brevity.
(ii) If $k>0$, then we say $\mathcal{F}$ is $\mathcal{P}$-sharp in $U$ if $F_{k}$ is $P_{k}$-sharp in $U$, and for any open set $U^{\prime} \subseteq U$ with $U^{\prime} \cap F_{k}, \mathcal{F} \upharpoonright k$ is $\mathcal{P} \upharpoonright k$-sharp in some open set $U^{\prime \prime} \subseteq U^{\prime}$.
Proposition 4.3. Suppose $\mathcal{F}=\left\langle F_{0}, \cdots, F_{k}\right\rangle$ is $\mathcal{P}$-sharp in $U$. Then for any $U^{\prime} \subseteq U$ with $U^{\prime} \cap F_{k} \neq \emptyset$, we have $\mathcal{F}$ is $\mathcal{P}$-sharp in $U^{\prime}$.

Proposition 4.4. Suppose $\mathcal{F}$ is $\mathcal{P}$-sharp in $U$. Then for any $m<\operatorname{lh}(\mathcal{F})$, $\mathcal{F} \upharpoonright m$ is $\mathcal{P} \upharpoonright m$-sharp in some open set $U^{\prime} \subseteq U$.

Let $\mathcal{P}=\left\langle P_{0}, \cdots, P_{k}\right\rangle, 0 \leq j \leq l$, and let $C \subseteq P_{j}$. We denote

$$
\mathcal{P} \backslash C=\left\langle P_{0}, \cdots, P_{j} \backslash C, \cdots, P_{k}\right\rangle
$$

Lemma 4.5. Let $\mathcal{F}=\left\langle F_{0}, \cdots, F_{k}\right\rangle, \mathcal{P}=\left\langle P_{0}, \cdots, P_{k}\right\rangle$. Suppose $\mathcal{F}$ is $P$ sharp in $U$. Let $0 \leq j \leq k$ and $\left(C_{l}\right)_{l<m}$ be a sequence of pairwise disjoint closed subsets of $P_{j}$. Then there exist at most one $l$ such that $\mathcal{F}$ is not $\mathcal{P} \backslash C_{l}$-sharp in any open set $U^{\prime} \subseteq U$.

Proof. We begin with $k=j=0$. Without loss of generality, suppose there exists an $l<m$, say, $l=0$, such that $F_{0}$ is not $P_{0} \backslash C_{0}$-sharp in $U$. Then there exists an open set $U_{0} \subseteq U$ with $U_{0} \cap F_{0} \neq \emptyset$ such that

$$
f \upharpoonright\left(f^{-1}\left(P_{0} \backslash C_{0}\right) \cap F_{0} \cap U_{0}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \Delta_{3}^{0}\right) .
$$

Assume for contradiction that there exists $l \neq 0$ such that $F_{0}$ is not $P_{0} \backslash C_{l^{-}}$ sharp in $U_{0}$, then there is an open set $U_{l} \subseteq U_{0}$ with $U_{l} \cap F_{0} \neq \emptyset$ such that

$$
f \upharpoonright\left(f^{-1}\left(P_{0} \backslash C_{l}\right) \cap F_{0} \cap U_{l}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \Delta_{3}^{0}\right) .
$$

Since $C_{0}$ and $C_{l}$ are disjoint closed subsets of $P_{0}$, Proposition 2.1 gives

$$
f \upharpoonright\left(f^{-1}\left(P_{0}\right) \cap F_{0} \cap U_{l}\right) \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right),
$$

contradicting that $F_{0}$ is $P_{0}$-sharp in $U$.
For $k>0$, assume that we have proved for all $k^{\prime}<k$.
Case 1. If $j=k$, since $F_{k}$ is $P_{k}$-sharp in $U$, from the arguments for $k=0$ above, we may assume that there is an open set $U_{0} \subseteq U$ with $U_{0} \cap F_{k} \neq \emptyset$
such that $F_{k}$ is $P_{k} \backslash C_{l}$-sharp in $U_{0}$ for any $l \neq 0$. It follows that $\mathcal{F}$ is $\mathcal{P} \backslash C_{l}$-sharp $U_{0}$ for any $l \neq 0$.

Case 2. If $j<k$, assume for contradiction that there are more than one $l$, say, $l=0,1$, such that $\mathcal{F}$ is not $\mathcal{P} \backslash C_{l}$-sharp in any open set $U^{\prime} \subseteq U$. Particularly, $\mathcal{F}$ is not $\mathcal{P} \backslash C_{0}$-sharp in $U$. Note that $F_{k}$ is $P_{k} \backslash C_{l}$-sharp in $U$ for any $l<m$, so there exists an $U_{0} \subseteq U$ with $U_{0} \cap F_{k} \neq \emptyset$ such that $\mathcal{F} \upharpoonright k$ is not $(\mathcal{P} \upharpoonright k) \backslash C_{0}$-sharp in any open set $U^{\prime} \subseteq U_{0}$. Similarly, we can find an open set $U_{1} \subseteq U_{0}$ with $U_{1} \cap F_{k} \neq \emptyset$ such that $\mathcal{F} \upharpoonright k$ is not $(\mathcal{P} \upharpoonright k) \backslash C_{1}$-sharp in any $U^{\prime} \subseteq U_{1}$. By Propositions 4.3 and 4.4, there is an open set $U^{*} \subseteq U_{1}$ such that $\mathcal{F} \upharpoonright k$ is $\mathcal{P}$-sharp in $U^{*}$, contradicting the induction hypothesis.

Theorem 4.6. Let $X$ be a Polish space, $Y$ a separable metrizable space, and let $f: X \rightarrow Y$ be of Baire class 1. If $f^{-1} \boldsymbol{\Sigma}_{3}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0}$, then $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$.

Proof. Assume for contradiction that $f \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$. We will define a continuous embedding $\psi: 2^{\omega} \rightarrow X$ and an $G_{\delta}$ set $G \subseteq Y$ such that $\psi^{-1}\left(f^{-1}(Y \backslash G)\right)=\Omega$. Thus $f^{-1}(G)$ is $\Pi_{3}^{0}$-complete subset of $X$, contradicting $f^{-1} \boldsymbol{\Sigma}_{3}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0}$.

It it well known that $Y$ is homeomorphic to a subspace of $\mathbb{R}^{\omega}$. Without loss of generality, we may assume $Y=\mathbb{R}^{\omega}$. Granting this assumption, we can fix a sequence of continuous functions $f_{n}: X \rightarrow Y$ pointwisely converging to $f$. Fix a compatible metric $d$ on $X$ with $d \leq 1$.

For $s \neq \emptyset$, let $\operatorname{lh}(s)=\ulcorner i, j\urcorner+1$. Now we redefine inheritors and innovators. We say $s$ is an inheritor if $j>0$ and $s(\ulcorner k, i+j-k\urcorner)=0$ for any $k \leq i$ (note: it was for any $k<i$ in the definition of inheritor in Theorem [3.6), otherwise we say $s$ is an innovator. Note that $s$ is always an innovator if $j=0$ or $s(\ulcorner i, j\urcorner)=1$.

We will inductively construct for each $s \in 2^{<\omega}$ an open set $V_{s}$ of $Y$, two closed sets $E_{s}, F_{s}$ of $X$, an open set $U_{s}$ of $X$, and a sequence of open sets $\left(U_{s}^{w}\right)_{s \preceq w \prec s \sim 0}$ of $X$ satisfying the following:
(0) $\operatorname{diam}\left(\overline{U_{s}}\right) \leq 2^{-\operatorname{lh}(s)}, U_{s \sim 0} \cap U_{s\urcorner 1}=\emptyset, \overline{U_{s\urcorner 0}}, \overline{U_{s\urcorner 1}} \subseteq U_{s}^{w}$;
(1) $F_{s \sim 1} \subseteq F_{s \neg 0}$;
(2) $\bar{V}_{s} \subseteq V_{\emptyset}$ and $F_{s} \subseteq E_{\emptyset}$ for any $s \neq \emptyset$;
(3) for any $s, t \neq \emptyset$ with $\operatorname{row}(s)=\operatorname{row}(t)$, we have $V_{s}=V_{t}$ or $\overline{V_{s}} \cap \overline{V_{t}}=\emptyset$;
(4) if $\operatorname{col}(s)>0$, then $\overline{V_{s\urcorner 0}}, \overline{V_{s \neg 1}} \subseteq V_{s}$ and $F_{s\urcorner 0} \subseteq F_{s}$;
(5) $E_{s} \subseteq F_{s}$;
(6) $E_{s} \cap U_{s}^{w} \neq \emptyset$ for each $w$;
(7) $U_{s}=U_{s}^{s}$, and $U_{s}^{w_{1}} \supseteq U_{s}^{w_{2}}$ for $w_{1} \preceq w_{2}$;
(8) if $s$ is an inheritor, then we have

$$
V_{s}=V_{v(s)}, \quad F_{s}=F_{v(s)}, \quad E_{s}=E_{u(s)}
$$

(9) if $s$ is an innovator, then $\overline{V_{s}} \cap \overline{V_{t}}=\emptyset$ for any $t$ with $t \prec s$ and $\operatorname{row}(t)=$ row $(s)$; furthermore, there exists $n \geq \operatorname{lh}(s)$ such that $f_{n}\left(U_{s}\right) \subseteq V_{s}$;
(10) if $s \neq \emptyset$, by letting $V_{s}^{-}= \begin{cases}V_{s \backslash(\operatorname{lh}(s)-1)}, & \operatorname{row}(s)>0, \\ V_{\emptyset}, & \operatorname{row}(s)=0,\end{cases}$

$$
\begin{gathered}
P_{s}^{r}=V_{s}^{-} \backslash \bigcup\left\{\overline{V_{t}}: t \preceq r, \operatorname{row}(t)=\operatorname{row}(s)\right\}, \\
\mathcal{P}_{s}^{r}=\left\langle P_{s \backslash(\operatorname{lh}(s)-\operatorname{row}(s))}^{r}, \cdots, P_{s}^{r}, V_{s}\right\rangle, \\
\mathcal{F}_{s}=\left\langle F_{s\lceil(\operatorname{lh}(s)-\operatorname{row}(s))}, \cdots, F_{s}, E_{s}\right\rangle,
\end{gathered}
$$

then for any $t \preceq s \prec t \wedge 0$, we have
(a) if $t^{\wedge} 0$ is an innovator, then $\mathcal{F}_{t}$ is $\mathcal{P}_{t}^{s}$-sharp in $U_{t}^{s}$;
(b) if $t^{\curvearrowright} 0$ is an inheritor, then $\mathcal{F}_{\left.u\left(t^{\prime}\right) 0\right)}$ is $\mathcal{P}_{u\left(t^{\prime} \sim 0\right)}^{s}$-sharp in $U_{t}^{s}$.

When we complete the construction, for any $z \in 2^{\omega}$, we set $\psi(z)$ to be the unique element of $\bigcap_{k} U_{z \mid k}$. From (0) and (7), $\psi$ is continuous embedding from $2^{\omega}$ to $X$. Put

$$
G_{m}=\bigcup_{\operatorname{row}(t)=m} V_{t}, \quad G=\bigcap_{m<\omega} G_{m} .
$$

If $z \in \Omega$, there exist $i_{0}<\omega$ and a strictly increasing sequence $j_{k}>0$ with $z\left(\left\ulcorner i_{0}, j_{k}\right\urcorner\right)=1$ for any $k<\omega$. Since $z \upharpoonright\left(\left\ulcorner i_{0}, j_{k}\right\urcorner+1\right)$ is an innovator, by (9), there is $n_{k}>\left\ulcorner i_{0}, j_{k}\right\urcorner$ such that $f_{n_{k}}(\psi(z)) \in V_{z \backslash\left(\left\ulcorner i_{0}, j_{k}\right\urcorner+1\right)}$. It follows from (9) that $f(\psi(z)) \notin V_{t}$ whenever $\operatorname{row}(t)=i_{0}$. Thus

$$
f(\psi(z)) \notin G_{i_{0}} \supseteq G .
$$

If $z \notin \Omega$, we show that $f(\psi(z)) \in G$. For any $m<\omega$, there exists $J_{m}<\omega$ such that $z(\ulcorner i, j\urcorner)=0$ for any $i \leq m$ and any $j>J_{m}$. So $z \upharpoonright(\ulcorner m, j\urcorner+1)$ is an inheritor for any $j>J_{m}$. Denote

$$
V_{m}=V_{z \backslash\left(\left\ulcorner m, J_{m}\right\urcorner+1\right)}, \quad u_{j}^{m}=u(z \upharpoonright(\ulcorner m, j\urcorner+1)) .
$$

By (8), we have $V_{m}=V_{z\lceil(\ulcorner m, j\urcorner+1)}$ for all $j>J_{m}$. Since all $u_{j}^{m}$ are innovators, by (9) we can find an $n_{j} \geq \operatorname{lh}\left(u_{j}^{m}\right)$ such that $f_{n_{j}}(\psi(z)) \in V_{u_{j}^{m}}$. By (4) and (8) we have $f_{n_{j}}(\psi(z)) \in V_{m}$ for all $j>J_{m}$. So $f(\psi(z)) \in \overline{V_{m}}$ for each $m$. Again by (4) we have $\overline{V_{m+1}} \subseteq V_{m}$ for any $m<\omega$. So

$$
f(\psi(z)) \in \overline{V_{m+1}} \subseteq V_{m} \subseteq G_{m}
$$

for all $m<\omega$. It follows that $f(\psi(z)) \in G$.
Now we turn to the construction.
First, set $D, P, h, \mathcal{B}_{P}$ as follows:
(i) $P=Y, D=X, h=f$;
(ii) $\mathcal{B}_{P}$ is a countable basis of $Y$.

Applying Lemma 4.1 with these $D, P, h, \mathcal{B}_{P}$, we get an open set $V$ of $Y$ and two closed sets $E \subseteq F$ of $X$. Then put

$$
V_{\emptyset}=V, \quad F_{\emptyset}=F, \quad E_{\emptyset}=E, \quad U_{\emptyset}=X .
$$

Secondly, assume that we have constructed $V_{t}, E_{t}, F_{t}, U_{t}$, and $U_{t}^{w}$ for $t, w \prec s^{\wedge} 0$. We will define for $s^{\wedge} 0$ and $s^{\wedge} 1$. We consider the following two cases:

Case 1. Assume $s\urcorner 0$ is an inheritor. Let $v=v\left(s^{\wedge} 0\right), u=u\left(s^{\wedge} 0\right)$. Put

$$
V_{s\urcorner 0}=V_{v}, \quad F_{s\urcorner 0}=F_{v}, \quad E_{s\urcorner 0}=E_{u} .
$$

Note that either $s=u$, or $s$ is also an inheritor with $u(s)=u$, so $E_{s}=E_{u}$. By (7), $E_{u} \cap U_{s}^{s_{M}} \neq \emptyset$, so we can define an open set $U_{s \neg 0}$ such that

$$
\overline{U_{s \vee 0}} \subseteq U_{s}^{s_{M}}, \quad U_{s^{\wedge} 0} \cap E_{u} \neq \emptyset, \quad \operatorname{diam}\left(\overline{U_{s \sim 0}}\right) \leq 2^{-(\operatorname{lh}(s)+1)} .
$$

We set $U_{s\urcorner 0}^{s \neg 0}=U_{s \sim 0}$ and $U_{t}^{s^{s} 0}=U_{t}^{s_{M}}$ for other $t$.
To check (0)-(10), the only nontrivial one is (10)(a) with $t=s^{\wedge} 0$. Note that, if $s^{\wedge} 0^{\wedge} 0$ is innovator, then $\operatorname{col}\left(s^{\wedge} 0^{\wedge} 0\right)=0$, i.e., $u=v$, so $\mathcal{P}_{s \vee 0}^{s^{\wedge} 0}=\mathcal{P}_{u}^{s_{M}}$ and $\mathcal{F}_{s^{\wedge}}=\mathcal{F}_{u}$. Since (10) holds for $s_{M}$, it holds for $s^{\wedge} 0$ too.

By shrinking, we may assume $E_{u} \cap\left(U_{s}^{s_{M}} \backslash \overline{U_{s \sim 0}}\right) \neq \emptyset$. By (10)(b) and Proposition 4.3, we see $\mathcal{F}_{u}$ is $\mathcal{P}_{u}^{s_{M}}$-sharp in $U_{s}^{s_{M}} \backslash \overline{U_{s \sim 0}}$. By Proposition 4.4. $\mathcal{F}_{u} \upharpoonright(\operatorname{row}(v)+1)$ is $\mathcal{P}_{u}^{s_{M}} \upharpoonright(\operatorname{row}(v)+1)$-sharp in some open set $U \subseteq\left(U_{s}^{s_{M}} \backslash\right.$ $\left.\overline{U_{s \sim 0}}\right)$. Thus $F_{v}$ is $P_{v}^{s_{M}-\text { sharp in } U \text {, and hence }}$

$$
f \upharpoonright\left(f^{-1}\left(P_{s}^{s_{M}}\right) \cap F_{v} \cap U\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

We inductively define $V^{l}, E^{l}$, and $F^{l}$ for each $l<\omega$. Denote $F^{-1}=F_{v}$. Assume that we have defined $V^{k}, E^{k}$, and $F^{k}$ for $k<l$. Set $D, P, h, \mathcal{B}_{P}$ as follows:
(i) $P=P_{s}^{s_{M}} \backslash \bigcup_{k<l} \overline{V^{k}}, D=F^{l-1} \cap U \cap f^{-1}(P), h=f \upharpoonright D$;
(ii) $\mathcal{B}_{P}$ is a countable basis of $P$ such that $\bar{V} \subseteq P$ for each $V \in \mathcal{B}_{P}$.

Applying Lemma 4.1 with these $D, P, h, \mathcal{B}_{P}$, we get an open set $V$ of $Y$ and two closed sets $E \subseteq F \subseteq \bar{D} \subseteq F^{l-1}$ with $E \cap U \supseteq E \cap D \neq \emptyset$. Denote $V^{l}=V, E^{l}=E$, and $F^{l}=F$. This complete the induction.

For $s \prec t \preceq s^{\wedge} 0$, if $t^{\wedge} 0$ is an innovator, it follows from (10)(a) that $\mathcal{F}_{t}$ is $\mathcal{P}_{t}^{s^{`} 0}$-sharp in $U_{t}^{s^{`}}$. By Lemma 4.5, we can find an natural number $L_{t}$ such that, for any $l \geq L_{t}, \mathcal{F}_{t}$ is $\mathcal{P}_{t}^{s^{\wedge} 0} \backslash \bar{V}^{l}$-sharp in some $U_{t}^{l} \subseteq U_{t}^{s^{\wedge} 0}$. If $t^{\wedge} 0$ is an inheritor, from (10)(b) and Lemma 4.5, we can also find an natural number $L_{t}$ such that, for any $l \geq L_{t}, \mathcal{F}_{u\left(t^{\wedge}\right)}$ is $\mathcal{P}_{t}^{s^{\wedge} 0} \backslash \bar{V}^{l}$-sharp in some open set $U_{t}^{l} \subseteq U_{t}^{s_{M}}$. Then we set

$$
L=\max \left\{L_{t}: s \prec t \preceq s^{\wedge} 0\right\}
$$

and $U_{t}^{s \wedge 1}=U_{t}^{L}$ for $t \preceq s^{\wedge} 0 \prec t^{\wedge} 0$, i.e., for $s \prec t \preceq s^{\wedge} 0$.
From Lemma 4.1 and $F^{L} \subseteq E_{s}$, we can see that $\left(\mathcal{F}_{s} \upharpoonright \operatorname{row}\left(s^{\wedge} 1\right)\right)^{\wedge} F^{L \wedge} E^{L}$ is $\left(\mathcal{P}_{s}^{s_{M}} \backslash \overline{V^{L}}\right)^{\wedge} V^{L}$-sharp in $U$.

Pick an $x \in\left(f^{-1}\left(V^{L}\right) \cap E^{L} \cap U\right)$. Since $f(x) \in V^{L}$, there is an $n>\operatorname{lh}(s)$ such that $f_{n}(x) \in V^{L}$. Then we can define an open set $U_{s^{\wedge 1}}$ such that

$$
\overline{U_{s \sim 1}} \subseteq U, \quad f_{n}\left(U_{s \sim 1}\right) \subseteq V^{L}, \quad x \in U_{s \sim 1}, \quad \operatorname{diam}\left(\overline{U_{s \sim 1}}\right) \leq 2^{-(\mathrm{lh}(s)+1)} .
$$

Then put

$$
V_{s\urcorner 1}=V^{L}, \quad E_{s\urcorner 1}=E^{L}, \quad F_{s\urcorner 1}=F^{L},
$$

and $U_{s\urcorner 1}^{s \wedge 1}=U_{s \wedge 1}$.

Case 2. Assume $s^{\wedge} 0$ is an innovator. Since $s \preceq s_{M} \prec s^{\wedge} 0$, by (10)(a), we have $\mathcal{F}_{s}$ is $\mathcal{P}_{s}^{s_{M}}$-sharp in $U_{s}^{s_{M}}$. Thus $E_{s}$ is $V_{s}$-sharp in $U_{s}^{s_{M}}$, and hence

$$
f \upharpoonright\left(f^{-1}\left(V_{s}\right) \cap E_{s} \cap U_{s}^{s_{M}}\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

Set $D, P, h, \mathcal{B}_{P}$ as follows:
(i) $P=V_{s}, D=E_{s} \cap U_{s}^{s_{M}} \cap f^{-1}(P), h=f \upharpoonright D$;
(ii) $\mathcal{B}_{P}$ is a countable basis of $P$ such that $\bar{V} \subseteq P$ for each $V \in \mathcal{B}_{P}$.

Applying Lemma 4.1 with these $D, P, h, \mathcal{B}_{P}$, we get an open set $V$ of $Y$ and two closed sets $E \subseteq F \subseteq \bar{D} \subseteq E_{s}$ with $E \cap U_{s}^{s_{M}} \supseteq E \cap D \neq \emptyset$. From Lemma 4.1 and $F \subseteq E_{s}$, we can see that $\left(\mathcal{F}_{s} \upharpoonright \operatorname{row}\left(s^{\wedge} 0\right)\right)^{\wedge} F^{\wedge} E$ is $\left(\mathcal{P}_{s}^{s_{M}} \backslash \bar{V}\right)^{\wedge} V$-sharp in $U_{s}^{s_{M}}$.

Pick an $x \in\left(f^{-1}(V) \cap E \cap U_{s}^{s_{M}}\right)$. Since $f(x) \in V$, there is an $n>\operatorname{lh}(s)$ such that $f_{n}(x) \in V$. Then we can an open set $U_{s \sim 0}$ such that

$$
\overline{U_{s\urcorner 0}} \subseteq U_{s}^{s_{M}}, \quad f_{n}\left(U_{s\urcorner 0}\right) \subseteq V, \quad x \in U_{s\urcorner 0}, \quad \operatorname{diam}\left(\overline{U_{s\urcorner 0}}\right) \leq 2^{-(\operatorname{lh}(s)+1)}
$$

Then put

$$
V_{s\urcorner 0}=V, \quad E_{s\urcorner 0}=E, \quad F_{s\urcorner 0}=F,
$$

and $U_{s\urcorner 0}^{s \frown 0}=U_{s\urcorner 0}$, and $U_{t}^{s\urcorner 0}=U_{t}^{s_{M}}$ for other $t$.
To check (0)-(10), it is trivial for $s=\emptyset$. For $s \neq \emptyset$, the only nontrivial clauses are (3), (9), and (10). Note that $\operatorname{row}\left(s^{\wedge} 0\right)>0$, so $V_{s \vee 0}^{-}=V_{s}$. Note also that either $s$ is also an innovator, or $\operatorname{col}\left(s^{\wedge} 0\right)=0$, i.e., $u(s)=v(s)$. In both cases, (4) and (9) imply that there is no $t \prec s^{\wedge} 0$ such that $\operatorname{row}(t)=$ $\operatorname{row}\left(s^{\wedge} 0\right)$ and $V_{t}^{-}=V_{s}$. So (3) and (9) hold. Therefore, $P_{s \sim 0}^{s^{\wedge}}=V_{s} \backslash \bar{V}$, thus $\mathcal{P}_{s\urcorner 0}^{s^{\wedge 0}}=\left(\mathcal{P}_{s}^{s_{M}} \backslash \bar{V}\right)^{\wedge} V$. Similarly, $\mathcal{P}_{t}^{s^{\wedge} 0}=\mathcal{P}_{t}^{s_{M}}$ and $\mathcal{P}_{u(t\urcorner 0)}^{s^{\wedge} 0}=\mathcal{P}_{u(t\urcorner 0)}^{s_{M}}$ for $t \prec s^{\wedge} 0 \prec t \wedge 0$. Since (10) holds for $s_{M}$, it holds for $s^{\wedge} 0$ too.

By shrinking, we may assume $F \cap\left(U_{s}^{s_{M}} \backslash \overline{U_{s \sim 0}}\right) \neq \emptyset$. By Lemma4.1,

$$
f \upharpoonright\left(f^{-1}\left(V_{s} \backslash \bar{V}\right) \cap F \cap\left(U_{s}^{s_{M}} \backslash \overline{U_{s \sim 0}}\right)\right) \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \Delta_{3}^{0}\right) .
$$

Now we define for $s \_1$ similar to the way in Case 1 .
Theorem 4.7. Let $X$ be a Polish space, $Y$ a separable metrizable space, and let $f: X \rightarrow Y$. Then

$$
f^{-1} \boldsymbol{\Sigma}_{3}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0} \Longleftrightarrow f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right) .
$$

Proof. The " $\Leftarrow$ " part is trivial, we only prove the " $\Rightarrow$ " part.
Since $f^{-1} \boldsymbol{\Sigma}_{3}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0}$ implies $f^{-1} \boldsymbol{\Sigma}_{2}^{0} \subseteq \boldsymbol{\Sigma}_{3}^{0}$, it follows from Theorem 3.6 that $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, i.e., there exists a sequence of $G_{\delta}$ set $X_{n}$ such that $\bigcup_{n} X_{n}=X$ and each $f \upharpoonright X_{n}$ is of Baire class 1. Then Theorem 4.6 gives $f \upharpoonright X_{n} \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$. Therefore, we have $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$.

Corollary 4.8. Let $X$ be a Polish space, $Y$ a separable metrizable space, and let $f: X \rightarrow Y$. If $f \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, then there exists a Cantor set $C \subseteq X$ such that $f \upharpoonright C \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$.

Proof. If $f \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, by Corollary 3.7] there exists a Cantor set $C \subseteq X$ such that $f \upharpoonright C \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$. It is clear that $f \upharpoonright C \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$.

If $f \in \operatorname{dec}\left(\boldsymbol{\Sigma}_{2}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$, i.e., there exists a sequence of $G_{\delta}$ set $X_{n}$ such that $\bigcup_{n} X_{n}=X$ and each $f \upharpoonright X_{n}$ is of Baire class 1, then there is some $X_{n}$ such that $f \upharpoonright X_{n} \notin \operatorname{dec}\left(\boldsymbol{\Sigma}_{1}^{0}, \boldsymbol{\Delta}_{3}^{0}\right)$. Let $\psi$ be the continuous embedding defined in Theorem 4.6. Put $C=\psi\left(2^{\omega}\right)$.

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