DECOMPOSING FUNCTIONS OF BAIRE CLASS 2 ON POLISH SPACES

LONGYUN DING, TAKAYUKI KIHARA, BRIAN SEMMES, AND JIAFEI ZHAO

ABSTRACT. We prove the Decomposability Conjecture for functions of Baire class 2 on a Polish space to a separable metrizable space. This partially answers an important open problem in descriptive set theory.

1. INTRODUCTION

In descriptive set theory, the study of decomposability of Borel functions originated by a famous question asked by Luzin around a century ago: Is every Borel function decomposable into countably many continuous functions? This question was answered negatively. Many counterexamples appeared in the literature (cf. [8, 10]) show that, even a function of Baire class 1 is not necessarily decomposable. Among these counterexamples, the Pawlikowski function $P: (\omega+1)^{\omega} \to \omega^{\omega}$ stands in an important position. Indeed, Solecki [15] proved that:

Let X, Y be separable metrizable spaces with X analytic, and let $f : X \to Y$ be of Baire class 1. Then f is not decomposable into countably many continuous functions iff $P \sqsubseteq f$, i.e., there exists embeddings $\phi : (\omega + 1)^{\omega} \to X$ and $\psi : \omega^{\omega} \to Y$ such that $\psi \circ P = f \circ \phi$.

Later, Pawlikowski and Sabok [13] generalized this theorem onto all Borel functions from an analytic space to a separable metrizable space. Motto Ros [11, Lemma 5.6] also gave an elegant proof for all functions of Baire class n with $n < \omega$.

A natural generalization of Luzin's question is to replace continuous functions with Σ_{γ}^{0} -measurable functions. We write $f \in \operatorname{dec}(\Sigma_{\gamma}^{0})$ if there exists a partition (X_{k}) of X with each $f \upharpoonright X_{k}$ is Σ_{γ}^{0} -measurable; and also write $f \in \operatorname{dec}(\Sigma_{\gamma}^{0}, \Delta_{\delta}^{0})$ if such a partition can be a sequence of Δ_{δ}^{0} subsets of

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X. It is trivial to see that, for $\delta \geq \gamma$, $f \in \operatorname{dec}(\Sigma^0_{\gamma}, \Delta^0_{\delta})$ implies the Σ^0_{δ} measurability of f. It is also well known that, for any Σ^0_{δ} -measurable function f with $\delta > \gamma$, we have $f \in \operatorname{dec}(\Sigma^0_{\gamma}) \iff f \in \operatorname{dec}(\Sigma^0_{\gamma}, \Delta^0_{\delta+1})$ (cf. [11, Proposition 4.5]).

A slightly more finer notion of Baire hierarchy was essentially introduced by Jayne [3] for studying the Banach space of functions of Baire class α . A function $f: X \to Y$ is called a $\Sigma_{\alpha,\beta}$ function (or more precisely denoted by $f^{-1}\Sigma^0_{\beta} \subseteq \Sigma^0_{\alpha}$) if the preimage $f^{-1}(A)$ is Σ^0_{α} in X for every Σ^0_{β} subset A of Y. The following theorem discovers a deep connection between this notion and decomposability:

Theorem 1.1 (Jayne-Rogers [4]). Let X, Y be separable metrizable spaces with X analytic, and let $f: X \to Y$. Then

$$f^{-1}\boldsymbol{\Sigma}_2^0 \subseteq \boldsymbol{\Sigma}_2^0 \iff f \in \operatorname{dec}(\boldsymbol{\Sigma}_1^0, \boldsymbol{\Delta}_2^0).$$

This theorem was generalized in [6] to the case that X is an absolute Souslin- \mathcal{F} set and Y is an arbitrary regular topological space.

It is conjectured that the Jayne-Rogers Theorem can be extended to all finite Borel ranks as follows:

The Decomposability Conjecture (cf. [1, 11, 13]). Let X, Y be separable metrizable spaces with X analytic, and let $f : X \to Y$. Then for $n \ge 2$ we have

$$f^{-1}\boldsymbol{\Sigma}_n^0 \subseteq \boldsymbol{\Sigma}_n^0 \iff f \in \operatorname{dec}(\boldsymbol{\Sigma}_1^0, \boldsymbol{\Delta}_n^0).$$

Furthermore, for $2 \le m \le n$ we have

$$f^{-1}\boldsymbol{\Sigma}_m^0 \subseteq \boldsymbol{\Sigma}_n^0 \iff f \in \operatorname{dec}(\boldsymbol{\Sigma}_{n-m+1}^0, \boldsymbol{\Delta}_n^0).$$

This conjecture was further generalized to The Full Decomposability Conjecture (see [2, Section 4]) which covers all infinite Borel ranks. Motto Ros presented an equivalent condition of the decomposability conjecture (see [11, Conjecture 6.1]). Another interesting equivalent condition with some extra restrictions on spaces and on relation between m, n, concerning computability on Borel codes from A to $f^{-1}(A)$, was given by Kihara in [9]. Most recently, Gregoriades-Kihara-Ng [2] proved

$$f^{-1}\Sigma_m^0 \subseteq \Sigma_n^0 \implies f \in \operatorname{dec}(\Sigma_{n-m+1}^0, \Delta_{n+1}^0).$$

It is clear that the case m = n = 2 in the decomposability conjecture is just the Jayne-Rogers Theorem. Remarkable progress is due to Semmes, the third author of this article. In his Ph.D. thesis [14], Semmes proved the case $m \leq n = 3$ for functions $f : \omega^{\omega} \to \omega^{\omega}$. In his proof, many kinds of games for characterizing Borel functions were widely applied. From the viewpoint of Jayne's work [3] in functional analysis, the zero-dimensionality constraint on Semmes' theorem was strongly desired to be removed. In this article, we generalize Semmes' theorem to arbitrary Polish spaces:

Theorem 1.2. The decomposability conjecture is true for the case that X is Polish space and $m \le n = 3$.

It is worth noting that in our proof, no game for Borel functions are involved. This is the key point that this proof can be extended to all Ploish spaces. This theorem consists of two cases: (a) m = 2, n = 3, and (b) m = n = 3. We will prove them in sections 3 and 4 respectively.

Following the outline of Semmes' proof, the proof appearing in this article was developed by the first and the forth authors. Almost at the same time, the second author independently gave a detailed exposition of Semmes' strategy. He also asserted that the use of games for Borel functions is misleading, and emphasized the use of finite injury priority argument instead. Soon after reading it, Motto Ros pointed out that the same argument in the second author's proof also works well, with some minor modifications, for arbitrary Polish spaces.

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2. Preliminaries

All topological spaces considered in this article are separable metrizable. For any subset A of a topological space X, we denote by \overline{A} the closure of A in X and denote $A^c = X \setminus A$ for brevity.

We recall some basic notations. A topological space is called a *Polish* space if it is separable and completely metrizable, and is called an *analytic* space if it is homeomorphic to an analytic subset of a Polish space. Given a separable metrizable space X, Borel sets of X can be analyzed into Borel hierarchy, consisting of Σ_{ξ}^{0} , Π_{ξ}^{0} subsets for $1 \leq \xi < \omega_{1}$. As usual, we denote $\Delta_{\xi}^{0} = \Sigma_{\xi}^{0} \cap \Pi_{\xi}^{0}$.

 $\boldsymbol{\Delta}^{0}_{\boldsymbol{\xi}} = \boldsymbol{\Sigma}^{0}_{\boldsymbol{\xi}} \cap \boldsymbol{\Pi}^{0}_{\boldsymbol{\xi}}.$ Let X, Y be two separable metrizable spaces, and $f: X \to Y$. We say f is $\boldsymbol{\Sigma}^{0}_{\alpha}$ -measurable if $f^{-1}(U) \in \boldsymbol{\Sigma}^{0}_{\alpha}$ for every open set $U \subseteq Y$. For the definition of the Baire classes of functions, one can see [7, (24.1)]. It is well known that a function is of Baire class $\boldsymbol{\xi}$ iff it is $\boldsymbol{\Sigma}^{0}_{\boldsymbol{\xi}+1}$ -measurable (cf. [7, (24.3)]).

In the section of introduction, we already presented notion of $\Sigma_{\alpha,\beta}$ functions, $f^{-1}\Sigma_{\beta}^{0} \subseteq \Sigma_{\alpha}^{0}$ and dec $(\Sigma_{\gamma}^{0}, \Delta_{\delta}^{0})$. The following proposition give some well known properties which will be used again and again in the rest of this article.

Proposition 2.1 (folklore). Let X, Y be two separable metrizable spaces, and let $f : X \to Y$. Then the following are equivalent:

- (i) $f \in \operatorname{dec}(\Sigma^0_{\gamma}, \Delta^0_{\delta}).$
- (ii) There exists a sequence (A_n) of Σ_{δ}^0 subsets with $X = \bigcup_n A_n$ such that every $f \upharpoonright A_n$ is Σ_{γ}^0 -measurable.

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(iii) There exists a sequence (A_n) of Σ^0_{δ} subsets with $X = \bigcup_n A_n$ such that every $f \upharpoonright A_n \in \operatorname{dec}(\Sigma^0_{\gamma}, \Delta^0_{\delta}).$

Proof. (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are trivial. We only prove (iii) \Rightarrow (i).

For every $n < \omega$, since $A_n \in \Sigma^0_{\delta}$, we can choose a sequence $(B_n^m)_{m < \omega}$ of Δ^0_{δ} sets such that $A_n = \bigcup_m B_n^m$. Moreover, since

$$f \upharpoonright A_n \in \operatorname{dec}(\mathbf{\Sigma}^0_{\gamma}, \mathbf{\Delta}^0_{\delta}),$$

there exist two sequences $(C_n^k)_{k<\omega}, (D_n^k)_{k<\omega}$ of Σ_{δ}^0 sets with

$$A_n \subseteq \bigcup_k C_n^k, \quad A_n \cap C_n^k = A_n \setminus D_n^k$$

such that each $f \upharpoonright (C_n^k \cap A_n)$ is Σ_{γ}^0 -measurable. Note that $B_n^m \cap C_n^k =$ $B_n^m \setminus D_n^k \in \mathbf{\Delta}_{\delta}^0$, and $f \upharpoonright (B_n^m \cap C_n^k)$ is $\mathbf{\Sigma}_{\gamma}^0$ -measurable for all $n, k, m < \omega$. Let $(K_l)_{l < \omega}$ be an enumeration of all $B_n^m \cap C_n^k$, $n, k, m < \omega$. Then

$$\bigcup_{l} K_{l} = \bigcup_{n,k,m} (B_{n}^{m} \cap C_{n}^{k}) = X.$$

For each $l < \omega$, put $K'_l = K_l \setminus (\bigcup_{i < l} K_i)$. Then the sequence $(K'_l)_{l < \omega}$ of Δ^0_{Δ} subsets witnesses that $f \in \operatorname{dec}(\Sigma^0_{\gamma}, \overline{\Delta}^0_{\delta})$.

3. The decomposability conjecture for m = 2, n = 3

We prove Theorem 1.2 for m = 2, n = 3 in this section, and for m = n = 3in the next section. The following lemma is the key tool for proving the main theorem of this section, just like the role of Lemma 4.3.3 in [14].

Lemma 3.1. Let X, P be two separable metrizable spaces, and let $D \subseteq X$, $h: D \to P$ a function of Baire class 2. Let \mathcal{B}_P be a countable topological basis of P, and for each $V \in \mathcal{B}_P$, let \mathcal{G}_V be a countable class of subsets of D such that

$$h^{-1}(V) = \bigcup \mathcal{G}_V.$$

If $h \notin \text{dec}(\mathbf{\Sigma}_2^0, \mathbf{\Delta}_3^0)$, then there exist $V \in \mathcal{B}_P, G \in \mathcal{G}_V$, and a closed set $F \subseteq \overline{D}$ satisfying:

(a) For any open set U with $F \cap U \neq \emptyset$,

$$h \upharpoonright (h^{-1}(\overline{V}^c) \cap F \cap U) \notin \operatorname{dec}(\boldsymbol{\Sigma}_2^0, \boldsymbol{\Delta}_3^0);$$

- (b) $G \cap F$ is dense in F;
- (c) $F \cap D \neq \emptyset$.

Proof. Let $\{U_k : k < \omega\}$ be a topological basis of X. For any $V \in \mathcal{B}_P$ and any closed subset $F \subseteq X$, we denote

$$\Gamma_V(F) = \{k < \omega : h \upharpoonright (h^{-1}(\overline{V}^c) \cap F \cap U_k) \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0)\},\$$
$$\Theta_V(F) = \{x \in F : \forall k < \omega(x \in U_k \Rightarrow k \notin \Gamma_V(F))\}.$$

It is trivial to see that $\Theta_V(F) \subseteq \overline{D}$ is closed.

For any $G \in \mathcal{G}_V$, we define closed set $F_{V,G}^{\alpha}$ for $\alpha < \omega_1$ as follows:

$$F_{V,G}^{0} = X,$$

$$F_{V,G}^{\alpha+1} = \overline{G \cap \Theta_V(F_{V,G}^{\alpha})},$$

$$F_{V,G}^{\lambda} = \bigcap_{\alpha < \lambda} F_{V,G}^{\alpha}, \quad \text{for limit ordinal } \lambda.$$

Since X is second countable, there exists a $\xi < \omega_1$ such that $F_{V,G}^{\alpha} = F_{V,G}^{\xi}$ for each V, G and $\alpha \geq \xi$.

If there exist $V \in \mathcal{B}_P, G \in \mathcal{G}_V$ such that $F_{V,G}^{\xi} \neq \emptyset$, then V, G, and $F = F_{V,G}^{\xi}$ fulfil clauses (a) and (b). Set U = X in (a), we can see (c) is also fulfilled.

Assume for contradiction that, for any $V \in \mathcal{B}_P, G \in \mathcal{G}_V$, we have $F_{V,G}^{\xi} = \emptyset$. For $\alpha < \xi$ and $k \in \Gamma_V(F_{V,G}^{\alpha})$, put

$$H^{\alpha}_{V,G,k} = F^{\alpha}_{V,G} \cap U_k.$$

Note that

$$h \upharpoonright (h^{-1}(\overline{V}^c) \cap H^{\alpha}_{V,G,k}) \in \operatorname{dec}(\Sigma^0_2, \Delta^0_3).$$

Now define a subset H of all x satisfying that, there exist $V_1, V_2 \in \mathcal{B}_P$ with $\overline{V_1} \cap \overline{V_2} = \emptyset$, and for i = 1, 2, there exist $G_i \in \mathcal{G}_{V_i}, \alpha_i < \xi$, and $k_i \in \Gamma_{V_i}(F_{V_i,G_i}^{\alpha_i})$ such that $x \in H_{V_1,G_1,k_1}^{\alpha_1} \cap H_{V_2,G_2,k_2}^{\alpha_2}$. Since

$$h \upharpoonright (h^{-1}(\overline{V_i}^c) \cap H^{\alpha_1}_{V_1,G_1,k_1} \cap H^{\alpha_2}_{V_2,G_2,k_2}) \in \operatorname{dec}(\Sigma^0_2, \Delta^0_3) \quad (i = 1, 2),$$

and $h^{-1}(\overline{V_i}^c)$ is Σ_3^0 in D for i = 1, 2, by Proposition 2.1, we have

$$h \upharpoonright (D \cap H^{\alpha_1}_{V_1,G_1,k_1} \cap H^{\alpha_2}_{V_2,G_2,k_2}) \in \operatorname{dec}(\boldsymbol{\Sigma}^0_2, \boldsymbol{\Delta}^0_3).$$

Therefore, $h \upharpoonright (D \cap H) \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$.

For any $x \in D$, $V \in \mathcal{B}_P$, and $G \in \mathcal{G}_V$ with $x \in G$, we claim that there exist $\alpha < \xi$ and $k \in \Gamma_V(F_{V,G}^{\alpha})$ such that $x \in H_{V,G,k}^{\alpha}$. There is unique $\alpha < \xi$ such that $x \in (F_{V,G}^{\alpha} \setminus F_{V,G}^{\alpha+1})$. Note that $x \notin (F \setminus \overline{G \cap F})$ for any closed set $F \subseteq X$, so $x \notin (\Theta_V(F_{V,G}^{\alpha}) \setminus F_{V,G}^{\alpha+1})$. From the definition of $\Theta_V(F_{V,G}^{\alpha})$, we can find a $k \in \Gamma_V(F_{V,G}^{\alpha})$ such that $x \in U_k$. Then we have $x \in F_{V,G}^{\alpha} \cap U_k = H_{V,G,k}^{\alpha}$.

In the end, we consider $h \upharpoonright (D \setminus H)$. First, for any $x \in (D \setminus H)$, if $x \in H^{\alpha}_{V,G,k}$ for some $V \in \mathcal{B}_P$, $G \in \mathcal{G}_V$, $\alpha < \xi$, and $k \in \Gamma_V(F^{\alpha}_{V,G})$, we claim that $h(x) \in \overline{V}$. If not, we can find a $V' \in \mathcal{B}_P$ such that $h(x) \in V'$ and $\overline{V} \cap \overline{V'} = \emptyset$. Since $h^{-1}(V') = \bigcup \mathcal{G}_{V'}$, we can find an $G' \in \mathcal{G}_{V'}$ such that $x \in G'$. Hence $x \in H^{\alpha'}_{V',G',k'}$ for some $\alpha' < \xi$ and $k' \in \Gamma_{V'}(F^{\alpha'}_{V',G'})$, contradicting $x \notin H$. Secondly, let d be a compatible metric on P. For any $n < \omega$, let

$$(V_m^n, G_m^n, k_m^n, \alpha_m^n)_{m < \omega}$$

be an enumeration of all (V, G, k, α) with diam $(\overline{V}) \leq 1/n, G \in \mathcal{G}_V, \alpha < \xi$, and $k \in \Gamma_V(F_{V,G}^{\alpha})$. Denote

$$H_m^n = H_{V_m^n, G_m^n, k_m^n}^{\alpha_m^n}.$$

For any $x \in D$, we can find a $V \in \mathcal{B}_P$ with diam $(\overline{V}) \leq 1/n$ such that $h(x) \in V$ and a $G \in \mathcal{G}_V$ with $x \in G$. Hence $x \in H^{\alpha}_{V,G,k}$ for some $\alpha < \xi$ and $k \in \Gamma_V(F^{\alpha}_{V,G})$. It follows that $D \subseteq \bigcup_m H^n_m$. Put $K^n_m = H^n_m \setminus \bigcup_{k < m} H^n_k$ for each m. Then $(K^n_m)_{m < \omega}$ is a sequence of pairwise disjoint Δ^0_2 sets. Fix a $y^n_m \in \overline{V^n_m}$ for each m. Define $g_n(x) = y^n_m$ for all $x \in K^n_m$. Then g_n is of Baire class 1. Furthermore, we have $d(g_n(x), h(x)) \leq 1/n$ for all $x \in (D \setminus H)$. So $(g_n \upharpoonright (D \setminus H))_{n < \omega}$ uniformly converges to $h \upharpoonright (D \setminus H)$. It follows that $h \upharpoonright (D \setminus H)$ is of Baire class 1 also (see [7, (24.4) i)]).

Note that H is an F_{σ} set from its definition. So $D \setminus H$ is G_{δ} in D, and hence $h \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$. A contradiction!

In the rest of this section, we fix X be a Polish space, Y a separable metrizable space, and $f: X \to Y$ a Σ_3^0 -measurable function.

Definition 3.2. Let $\mathcal{F} = \langle F_0, \cdots, F_k \rangle$ be a finite sequence of closed sets of X with $F_0 \supseteq \cdots \supseteq F_k$, U an open subset of X, and let $P \subseteq Y$.

(i) If k = 0, i.e., $\mathcal{F} = \langle F_0 \rangle$, then we say \mathcal{F} is *P*-sharp in *U* if $U \cap F_0 \neq \emptyset$, and for any open set $U' \subseteq U$ with $U' \cap F_0 \neq \emptyset$, we have

$$f \upharpoonright (f^{-1}(P) \cap F_0 \cap U') \notin \operatorname{dec}(\Sigma_2^0, \Delta_3^0).$$

We also say F_0 itself is *P*-sharp in *U* for brevity.

(ii) If k > 0, then we say \mathcal{F} is *P*-sharp in *U* if F_k is *P*-sharp in *U*, and for any open set $U' \subseteq U$ with $U' \cap F_k \neq \emptyset$, $\mathcal{F} \upharpoonright k$ is *P*-sharp in some open set $U'' \subseteq U'$.

A similar notion named δ - σ -good was presented in [14]. The following propositions are trivial, we omit the proofs.

Proposition 3.3. Suppose $\mathcal{F} = \langle F_0, \cdots, F_k \rangle$ is *P*-sharp in *U*. Then for any open set $U' \subseteq U$ with $U' \cap F_k \neq \emptyset$, we have \mathcal{F} is *P*-sharp in *U'*.

Proposition 3.4. Suppose \mathcal{F} is *P*-sharp in *U*. Then for any $m < \mathrm{lh}(\mathcal{F})$, $\mathcal{F} \upharpoonright m$ is *P*-sharp in some open set $U' \subseteq U$.

The following lemma is modified from [14, Lemma 4.3.6].

Lemma 3.5. Suppose $\mathcal{F} = \langle F_0, \cdots, F_k \rangle$ is *P*-sharp in *U*. Let $(C_l)_{l < m}$ be a sequence of pairwise disjoint closed subsets of *P*. Then there exist at most k + 1 many *l* such that \mathcal{F} is not $P \setminus C_l$ -sharp in any open set $U' \subseteq U$.

Proof. We begin with k = 0. Without loss of generality, suppose there exists an l < m, say, l = 0, such that F_0 is not $P \setminus C_0$ -sharp in U. Then there exists an open set $U_0 \subseteq U$ with $U_0 \cap F_0 \neq \emptyset$ such that

$$f \upharpoonright (f^{-1}(P \setminus C_0) \cap F_0 \cap U_0) \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0).$$

Assume for contradiction that there exists $l \neq 0$ such that F_0 is not $P \setminus C_l$ sharp in U_0 , then there is an open set $U_l \subseteq U_0$ with $U_l \cap F_0 \neq \emptyset$ such that

$$f \upharpoonright (f^{-1}(P \setminus C_l) \cap F_0 \cap U_l) \in \operatorname{dec}(\boldsymbol{\Sigma}_2^0, \boldsymbol{\Delta}_3^0).$$

Since C_0 and C_l are disjoint closed subsets of P, Proposition 2.1 gives

$$f \upharpoonright (f^{-1}(P) \cap F_0 \cap U_l) \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0),$$

contradicting that F_0 is *P*-sharp in *U*.

For k > 0, assume that we have proved for all k' < k. Since F_k is P-sharp in U, from the arguments for k = 0 above, we may assume that there is an open set $U_0 \subseteq U$ with $U_0 \cap F_k \neq \emptyset$ such that F_k is $P \setminus C_l$ -sharp in U_0 for any $l \neq 0$. Assume for contradiction that there are k + 1 many $l \neq 0$, say, $l = 1, \dots, k, k + 1$, such that \mathcal{F} is not $P \setminus C_l$ -sharp in any open set $U' \subseteq U_0$. Particularly, \mathcal{F} is not $P \setminus C_1$ -sharp in U_0 , so there exists an open set $U_1 \subseteq U_0$ with $U_1 \cap F_k \neq \emptyset$ such that $\mathcal{F} \upharpoonright k$ is not $P \setminus C_1$ -sharp in any open set $U' \subseteq U_1$. Similarly, we can find a sequence of open sets $U_{k+1} \subseteq U_k \subseteq \dots \subseteq U_1 \subseteq U_0$ such that $U_l \cap F_k \neq \emptyset$ and $\mathcal{F} \upharpoonright k$ is not $P \setminus C_l$ -sharp in any $U' \subseteq U_l$ for $0 < l \le k+1$. By Definition of P-sharp, there is a open set $U^* \subseteq U_{k+1}$ such that $\mathcal{F} \upharpoonright k$ is P-sharp in U^* , contradicting the induction hypothesis. \Box

Let $\lceil \cdot, \cdot \rceil$ be the bijection: $\omega \times \omega \to \omega$ as following:

Denote

$$\Omega = \{ z \in 2^{\omega} : \exists i \exists^{\infty} j (z(\lceil i, j \rceil) = 1) \}.$$

It is well known that Ω is Σ_3^0 -complete subset of 2^{ω} .

For any $z \in 2^{\omega}$ and $l < \omega$, we call sequence

$$z \upharpoonright (\ulcorner 0, l\urcorner + 1), \ z \upharpoonright (\ulcorner 1, l - 1\urcorner + 1), \ \cdots, \ z \upharpoonright (\ulcorner l, 0\urcorner + 1)$$

the *l*-th **diagonal** of *z*, and call $z \upharpoonright (\ulcornerl, 0\urcorner + 1)$ the end of *l*-th diagonal. For $s \in 2^{<\omega}$, we denote $\ln(s) = i$ the length of *s*. If $s \subseteq z$ and $\ln(s) = \ulcorneri, j\urcorner + 1$, then *s* is in (i + j)-th diagonal. Moreover, the *l*-th diagonal of *z* is also named the *l*-th diagonal of *s* when $\ulcornerl, 0\urcorner < \ln(s)$.

For $s \neq \emptyset$, let $\ln(s) = \lceil i, j \rceil + 1$. We denote $\operatorname{row}(s) = i, \operatorname{col}(s) = j$. If i+j > 0, we call $s \upharpoonright (\lceil i+j-1, 0 \rceil + 1)$ the end of the last diagonal of s, denoted by u(s). If j > 0, we call $s \upharpoonright (\lceil i, j-1 \rceil + 1)$ the left neighbor of s, denoted by v(s).

For proving the following theorem, we need an order \leq on $2^{<\omega}$ define by

$$t \leq s \iff \ln(t) < \ln(s) \text{ or } (\ln(t) = \ln(s), t \leq_{\text{lex}} s),$$

where \leq_{lex} is the usual lexicographical order. We also denote $t \prec s$ when $t \preceq s$ but not t = s. Denote

$$s_M = \max_{\prec} \{t : t \prec s^{\frown} 0\}.$$

Theorem 3.6. Let X be a Polish space, Y a separable metrizable space, and let $f: X \to Y$. Then

$$f^{-1}\Sigma_2^0 \subseteq \Sigma_3^0 \iff f \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0).$$

Proof. The " \Leftarrow " part is trivial, we only prove the " \Rightarrow " part.

Assume for contradiction that $f \notin \text{dec}(\Sigma_2^0, \Delta_3^0)$. We will define a continuous embedding $\psi : 2^{\omega} \to X$ and an open set $O \subseteq Y$ such that

$$\psi^{-1}(f^{-1}(O)) = \Omega.$$

Thus $f^{-1}(Y \setminus O)$ is Π_3^0 -complete subset of X, contradicting $f^{-1}\Sigma_2^0 \subseteq \Sigma_3^0$.

For any open set $V \subseteq Y$, since $f^{-1}(V)$ is Σ_3^0 , we can fix a system of open set $W_n^m(V) \subseteq X$ $(m, n < \omega)$ with

$$f^{-1}(V) = \bigcup_{m \in \mathbb{N}} \bigcap_{n} W_n^m(V).$$

Denote $G^m(V) = \bigcap_n W_n^m(V)$. For $G = G^m(V)$, we also denote $W_n(G) = W_n^m(V)$.

For $s \in 2^{<\omega}$ with $s \neq \emptyset$, let $\ln(s) = \lceil i, j \rceil + 1$. We say s is an **inheritor** if j > 0 and $s(\lceil k, i + j - k \rceil) = 0$ for any k < i, otherwise we say s is an **innovator**. Note that s is always an inheritor if i = 0, j > 0, and is always an innovator if j = 0.

Fix a compatible metric d on X with $d \leq 1$. We will inductively construct, for each $s \in 2^{<\omega}$, an open set V_s of Y, a G_{δ} set G_s of X, a closed set F_s of X, an open set U_s of X, and a sequence of open sets $(U_s^w)_{s \leq w \leq s^0}$ of Xsatisfying the following:

- (0) diam $(\overline{U_s}) \leq 2^{-\ln(s)}, U_{s \cap 0} \cap U_{s \cap 1} = \emptyset, \overline{U_{s \cap 0}} \subseteq U_s^w, \overline{U_{s \cap 1}} \subseteq U_s^w;$
- (1) $V_{s^{\frown}0} = V_{s^{\frown}1}, \ G_{s^{\frown}0} = G_{s^{\frown}1}, \ F_{s^{\frown}0} = \underline{F_{s^{\frown}1}};$
- (2) for $s, t \in 2^{<\omega}$, we have $V_s = V_t$ or $\overline{V_s} \cap \overline{V_t} = \emptyset$;
- (3) there exist $m < \omega$ such that $G_s = G^m(V_s)$;
- (4) if $\operatorname{col}(s) > 0$, then $F_{s \cap 0} = F_{s \cap 1} \subseteq F_s$;
- (5) $F_s \cap U_s^w \neq \emptyset$ for each w;
- (6) $U_s = U_s^s$, and $U_s^{w_1} \supseteq U_s^{w_2}$ for $w_1 \preceq w_2$;
- (7) $G_s \cap F_s$ is dense in F_s ;
- (8) if s is an inheritor, then we have

$$V_s = V_{v(s)}, \quad G_s = G_{v(s)}, \quad F_s = F_{v(s)};$$

furthermore, (a) if $s \cap 0$ is also an inheritor, then $U_s^w \cap F_{u(s)} \neq \emptyset$ for each w; (b) if $s \cap 0$ is an innovator, then $U_s \subseteq W_n(G_s)$ for all $n < \mathrm{lh}(s)$;

- (9) if s is an innovator, then $U_s \cap F_s \cap G^m(V_t) = \emptyset$ for all $m < \ln(s)$ and all $\ln(t) < \ln(s)$;
- (10) by letting $P_s = Y \setminus \bigcup_{t \prec s} \overline{V_t}$,

$$\mathcal{F}_s = \langle F_{s \upharpoonright (\ln(s) - \operatorname{row}(s))}, \cdots, F_{s \upharpoonright (\ln(s) - 1)}, F_s \rangle_{s}$$

for $t \leq s < t^0$, we have

- (a) if t^0 is an innovator, then \mathcal{F}_t is P_s -sharp in U_t^s ;
- (b) if t^{0} is an inheritor, then $\mathcal{F}_{u(t^{0})}$ is P_{s} -sharp in U_{t}^{s} .

When we complete the construction, for any $z \in 2^{\omega}$, we set $\psi(z)$ to be the unique element of $\bigcap_k U_{z \mid k}$. From (0) and (6), we see that ψ is a continuous embedding from 2^{ω} to X. Put

$$O = \bigcup_{t \in 2^{<\omega}} V_t.$$

If $z \in \Omega$, let i_0 be the least *i* such that there are infinitely many *j* with $z(\lceil i, j \rceil) = 1$. Then there is $J_0 < \omega$ such that $z(\lceil i, j \rceil) = 0$ for all $i < i_0$ and all $j > J_0$. Hence for any $j > J_0$, $z \upharpoonright (\lceil i_0, j \rceil + 1)$ is an inheritor. Denote

$$V = V_{z \upharpoonright (\ulcorner i_0, J_0 \urcorner + 1)}, \quad G = G_{z \upharpoonright (\ulcorner i_0, J_0 \urcorner + 1)}.$$

By (8), we have $V = V_{z \upharpoonright (\ulcorner i_0, j \urcorner + 1)}$ and $G = G_{z \upharpoonright (\ulcorner i_0, j \urcorner + 1)}$ for all $j \ge J_0$. If $j > J_0$ with $z(\ulcorner i_0, j \urcorner) = 1$, by (8)(b), we have $\psi(z) \in W_n(G)$ for all $n \le \ulcorner i_0, j \urcorner$. Since there is infinitely many such j, we have $\psi(z) \in G \subseteq f^{-1}(V)$. Therefore, $f(\psi(z)) \in V \subseteq O$.

If $z \notin \Omega$, we show that $f(\psi(z)) \notin O$. If not, there exits t_1 and m_1 such that $\psi(z) \in G^{m_1}(V_{t_1})$. Fix an $i_1 > \max\{m_1, \ln(t_1)\}$. Since $z \notin \Omega$, for large enough j, we have $z(\lceil i, j \rceil) = 0$ for all $i < i_1$, so $z \upharpoonright (\lceil i_1, j \rceil + 1)$ is an inheritor. Let J_1 be the largest j such that $z \upharpoonright (\lceil i_1, j \rceil + 1)$ is an innovator. Denote

$$F = F_{z \upharpoonright (\lceil i_1, J_1 \rceil + 1)}.$$

By (8), we have $F = F_{z \upharpoonright (\lceil i_1, j \rceil + 1)}$ for all $j \ge J_1$. From (5) and (6), we see that $F \cap U_{z \upharpoonright (\lceil i_1, j \rceil + 1))} \neq \emptyset$ for any $j > J_1$, and hence $\psi(z) \in F$. It follows from (9) that $F \cap U_{z \upharpoonright (\lceil i_1, J_1 \rceil + 1)} \cap G^{m_1}(V_{t_1}) = \emptyset$. Thus $\psi(z) \notin G^{m_1}(V_{t_1})$. A contradiction!

Now we turn to the construction.

First, set $D, P, h, \mathcal{B}_P, \mathcal{G}_V$ as follows:

- (i) P = Y, D = X, and h = f;
- (ii) \mathcal{B}_P is a countable basis of Y;
- (iii) for each $V \in \mathcal{B}_P$, let $\mathcal{G}_V = \{G^m(V) : m < \omega\}$.

Applying Lemma 3.1 with these $D, P, h, \mathcal{B}_P, \mathcal{G}_V$, we get an open set $V \subseteq Y$, a G_{δ} set $G \in \mathcal{G}_V$, and a non-empty closed set $F \subseteq X$. Then put

$$V_{\emptyset} = V, \quad G_{\emptyset} = G, \quad F_{\emptyset} = F, \quad U_{\emptyset} = X.$$

Secondly, assume that we have constructed V_t, G_t, F_t, U_t , and U_t^w for $t, w \prec s \uparrow 0$. We will define for $s \uparrow 0$ and $s \uparrow 1$. We consider the following two cases:

Case 1. Assume $s \ 0$ is an inheritor. Let $v = v(s \ i), u = u(s \ i)$. For i = 0, 1, put

$$V_{s^\frown i} = V_v, \quad G_{s^\frown i} = G_v, \quad F_{s^\frown i} = F_v.$$

Note that s = u or s is also an inheritor with u(s) = u. By (5) and (8)(a), $U_s^{s_M} \cap F_u \neq \emptyset$.

Subcase 1.1. If $col(s^0^0) > 0$, then s^0^0 is still an inheritor. We can find a U_{s^0} such that

$$\overline{U_{s^{\frown}0}} \subseteq U_s^{s_M}, \quad U_{s^{\frown}0} \cap F_u \neq \emptyset, \quad \mathrm{diam}(\overline{U_{s^{\frown}0}}) \leq 2^{-(\mathrm{lh}(s)+1)}.$$

By (10)(b), we see \mathcal{F}_u is P_{s_M} -sharp in $U_s^{s_M}$. By Proposition 3.4, \mathcal{F}_v is P_{s_M} -sharp in some open set $U \subseteq U_s^{s_M}$, and hence $U \cap F_v \neq \emptyset$. Denote $W = \bigcap_{n \leq \mathrm{lh}(s)} W_n(G_v)$. Since $W \supseteq G_v$, by (7) we have $W \cap F_v$ is open dense in F_v , and hence $W \cap U \cap F_v \neq \emptyset$. We can find $U_{s \cap 1}$ such that

$$\overline{U_{s^{\uparrow}1}} \subseteq U \cap W, \quad U_{s^{\uparrow}1} \cap F_v \neq \emptyset, \quad \operatorname{diam}(\overline{U_{s^{\uparrow}1}}) \le 2^{-(\operatorname{lh}(s)+1)}.$$

By shrinking we may assume that $U_{s^{\frown}0} \cap U_{s^{\frown}1} = \emptyset$. For i = 0, 1, we set $U_{s^{\frown}0}^{s^{\frown}i} = U_{s^{\frown}0}, U_{s^{\frown}1}^{s^{\frown}1} = U_{s^{\frown}1}$, and for other t, set $U_t^{s^{\frown}i} = U_t^{s_M}$.

Subcase 1.2. If $\operatorname{col}(s^{\circ}0^{\circ}0) = 0$, then $s^{\circ}0^{\circ}0$ is an innovator. We define $U_{s^{\circ}i}$ for i = 0, 1 similar to $U_{s^{\circ}1}$ in Subcase 1.1 with one more requirement $U_{s^{\circ}0} \cap U_{s^{\circ}1} = \emptyset$.

It is trivial to check clauses (0)–(9). Note that $P_{s^{\uparrow}0} = P_{s^{\uparrow}1} = P_{s_M}$ and $\mathcal{F}_{s^{\uparrow}0} = \mathcal{F}_{s^{\uparrow}1} = \mathcal{F}_v$. Since (10) holds for s_M , it also holds for $s^{\uparrow}0$ and $s^{\uparrow}1$. Case 2. Assume $s^{\uparrow}0$ is an innovator. We inductively define V^l, G^l, F^l and U^l for each $l < \omega$ as the following:

Since $s \leq s_M \prec s^{\uparrow}0$, by (10)(a), we have \mathcal{F}_s is P_{s_M} -sharp in $U_s^{s_M}$. So

 $f \upharpoonright (f^{-1}(P_{s_M}) \cap F_s \cap U_s^{s_M}) \notin \operatorname{dec}(\boldsymbol{\Sigma}_2^0, \boldsymbol{\Delta}_3^0).$

Denote $F^{-1} = F_s, U^{-1} = U_s^{s_M}$. Assume that we have defined V^k, G^k, F^k and p^k for k < l. Set $D, P, h, \mathcal{B}_P, \mathcal{G}_V$ as follows:

(i) $P = P_{s_M} \setminus \bigcup_{k < l} \overline{V^k}, D = f^{-1}(P) \cap F^{l-1} \cap U^{l-1}, \text{ and } h = f \upharpoonright D;$

(ii) \mathcal{B}_P is a countable basis of P such that $\overline{V} \subseteq P$ for each $V \in \mathcal{B}_P$;

(iii) for each $V \in \mathcal{B}_P$, let $\mathcal{G}_V = \{D \cap G^m(V) : m < \omega\}.$

Applying Lemma 3.1 with these $D, P, h, \mathcal{B}_P, \mathcal{G}_V$, we get an open set $V \subseteq Y$, a G_{δ} set $G = G^m(V)$ for some $m < \omega$, and a closed set $F \subseteq \overline{D} \subseteq F^{l-1} \subseteq F_s$ with $F \cap U^{l-1} \supseteq F \cap D \neq \emptyset$. Denote $V^l = V, G^l = G, F^l = F$. If $\mathcal{F}_s^{\frown} F^l$ is $P_{s_M} \setminus \overline{V^l}$ -sharp in U^{l-1} , set $U^l = U^{l-1}$. Otherwise, since Lemma 3.1 implies that F^l is $P_{s_M} \setminus \overline{V^l}$ -sharp in U^{l-1} , we can find an open set $U^l \subseteq U^{l-1}$ with $U^l \cap F^l \neq \emptyset$ such that \mathcal{F}_s is not $P_{s_M} \setminus \overline{V^l}$ -sharp in any open set $U \subseteq U^l$. This complete the induction.

For $s \prec t \preceq s_M$, if $t \uparrow 0$ is an innovator, it follows from (10)(a) that \mathcal{F}_t is P_{s_M} -sharp in $U_t^{s_M}$. By Lemma 3.5, we can find a natural number L_t such that, for any $l \ge L_t$, \mathcal{F}_t is $P_{s_M} \setminus \overline{V^l}$ -sharp in some open set $U_t^l \subseteq U_t^{s_M}$. If $t \uparrow 0$ is an inheritor, from (10)(b) and Lemma 3.5, we can also find a natural number L_t such that, for any $l \ge L_t$, $\mathcal{F}_{u(t \uparrow 0)}$ is $P_{s_M} \setminus \overline{V^l}$ -sharp in some open set $U_t^l \subseteq U_t^{s_M}$. If $t \uparrow 0$ is an inheritor, from (10)(b) and Lemma 3.5, we can also find a natural number L_t such that, for any $l \ge L_t$, $\mathcal{F}_{u(t \uparrow 0)}$ is $P_{s_M} \setminus \overline{V^l}$ -sharp in some open set $U_t^l \subseteq U_t^{s_M}$. Moreover, assume for contradiction that there exist $l_0 < \cdots < l_m$ with $m = \ln(\mathcal{F}_s)$ such that \mathcal{F}_s is not $P_{s_M} \setminus \overline{V^{l_j}}$ -sharp in any

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open set $U \subseteq U^{l_j}$ for $j \leq m$. This contradicts Lemma 3.5, because $U^{l_m} \subseteq$ $U_s^{s_M}$ and $U^{l_m} \cap F^{l_m} \neq \emptyset$ implies that \mathcal{F}_s is P_{s_M} -sharp in U^{l_m} . Therefore, comparing with the definition of U^l , we can find an natural number L' such that, for any $l \geq L'$, $\mathcal{F}_s^{\frown} F^l$ is $P_{s_M} \setminus \overline{V^l}$ -sharp in U^l . Then we set

$$L = \max\{L', L_t : s \prec t \preceq s_M\}$$

and $U_t^{s^{\uparrow 0}} = U_t^{s^{\uparrow 1}} = U_t^L$ for $t \prec s^{\uparrow 0} \prec t^{\uparrow 0}$, i.e., for $s \prec t \preceq s_M$. In the end, denote

$$A = \bigcup_{\mathrm{lh}(t) \le \mathrm{lh}(s)} \bigcup_{m \le \mathrm{lh}(s)} G^m(V_t).$$

Then A is G_{δ} set. By (3) and $V^L \subseteq P_{s_M}$, we have $G^L \cap A = \emptyset$. It follows from Lemma 3.1 that $G^L \cap F^L$ is dense in F^L , so $A \cap F^L$ is nowhere dense in F^L . We can find two open sets $U_{s \cap 0}$ and $U_{s \cap 1}$ such that $U_{s \cap 0} \cap U_{s \cap 1} = \emptyset$, and for i = 0, 1, we have

$$\overline{U_{s^{\frown}i}} \subseteq U^L, \quad U_{s^{\frown}i} \cap F^L \neq \emptyset, \quad U_{s^{\frown}i} \cap F^L \cap A = \emptyset, \quad \operatorname{diam}(\overline{U_{s^{\frown}i}}) \le 2^{-(\operatorname{lh}(s)+1)}.$$
 Now put

Now put

$$V_{s^{\frown}i} = V^L, \quad G_{s^{\frown}i} = G^L, \quad F_{s^{\frown}i} = F^L,$$
 and $U_{s^{\frown}0}^{s^{\frown}i} = U_{s^{\frown}0}, U_{s^{\frown}1}^{s^{\frown}1} = U_{s^{\frown}1}.$

Corollary 3.7. Let X be a Polish space, Y a separable metrizable space, and let $f: X \to Y$. If $f \notin \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$, then there exists a Cantor set $C \subseteq X$ such that $f \upharpoonright C \notin \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$.

Proof. Let ψ be the continuous embedding defined in Theorem 3.6. Put $C = \psi(2^{\omega}).$

4. The decomposability conjecture for m = n = 3

Before proving Theorem 1.2 for m = n = 3, we prove a known result first: for functions of Baire class 1,

$$f^{-1}\Sigma_3^0 \subseteq \Sigma_3^0 \Rightarrow f \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0).$$

This is an easy corollary of Solecki's theorem (see [15, Theorem 4.1]), since $f \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0) \iff f \in \operatorname{dec}(\Sigma_1^0)$ and $P^{-1}\Sigma_3^0 \not\subseteq \Sigma_3^0$. Furthermore, this result is also a special case of [13, Corollary 1.2], [11, Corollary 5.11], or [2, Theorem 1.1]. In order to show a completely different method of proof, we present a direct proof which follows the same idea as in the previous section. The readers can skip directly to Theorem 4.7.

Lemma 4.1. Let X, P be two separable metrizable spaces, and let $D \subseteq X$, and $h: D \to P$ a function of Baire class 1. Let \mathcal{B}_P be a countable topological basis of P. If $h \notin \text{dec}(\Sigma_1^0, \Delta_3^0)$, then there exist a $V \in \mathcal{B}_P$ and two closed sets $E \subseteq F \subseteq \overline{D}$ satisfying:

(a) for any open set U with $E \cap U \neq \emptyset$, we have

$$h \upharpoonright (h^{-1}(V) \cap U \cap E) \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0);$$

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(b) for any open set U with $F \cap U \neq \emptyset$, we have

$$h \upharpoonright (h^{-1}(\overline{V}^c) \cap F \cap U) \notin \operatorname{dec}(\mathbf{\Sigma}_1^0, \mathbf{\Delta}_3^0);$$

(c) $E \cap D \neq \emptyset$.

Proof. Let $\{U_k : k < \omega\}$ be a topological basis of X. For any $B \in \mathcal{B}_P$, we denote

$$F^B = \{ x \in X : \forall k (x \in U_k \Rightarrow h \upharpoonright (h^{-1}(B^c) \cap U_k) \notin \operatorname{dec}(\boldsymbol{\Sigma}_1^0, \boldsymbol{\Delta}_3^0)) \}.$$

It is trivial to see that

- (i) F^B is closed,
- (i) $h \upharpoonright (h^{-1}(B^c) \setminus F^B) \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$, and (iii) for any open set U with $F^B \cap U \neq \emptyset$,

$$h \upharpoonright (h^{-1}(B^c) \cap F^B \cap U) \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0).$$

Assume for contradiction that, $h \upharpoonright (h^{-1}(B) \cap F^B) \in dec(\Sigma_1^0, \Delta_3^0)$ for any $B \in \mathcal{B}_P$. We denote

$$H_1 = \bigcup_{B \in \mathcal{B}_P} (h^{-1}(B) \cap F^B),$$
$$H_2 = \bigcup_{B \in \mathcal{B}_P} (h^{-1}(B^c) \setminus F^B).$$

It is straightforward to check that, $h \upharpoonright H_i \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$ for i = 1, 2.

Denote $H_3 = D \setminus (H_1 \cup H_2)$. For any $x \in H_3$ and any $B \in \mathcal{B}_P$, we have

$$h(x) \in B \Rightarrow x \in h^{-1}(B) \Rightarrow x \notin F^B,$$

$$h(x) \notin B \Rightarrow x \in h^{-1}(B^c) \Rightarrow x \in F^B.$$

So $h \upharpoonright H_3$ is continuous.

Let $Y \supseteq Y$ be a Polish space. By Kuratowski's theorem (cf. [7, (3.8)]), there is a G_{δ} set $G \supseteq H_3$ and a continuous function $g: G \to \tilde{Y}$ such that $g \upharpoonright H_3 = h \upharpoonright H_3$. Put $H = \{x \in D \cap G : h(x) = g(x)\}$. Since h is of Baire class 1, we see H is G_{δ} subset of D and $H_1 \cup H_2 \cup H = D$. Note that H_1 is F_{σ} subset of D and H_2 is Σ_3^0 subset of D. It follows that $h \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$. A contradiction!

Therefore, there exists a $B \in \mathcal{B}_P$ such that

$$h \upharpoonright (h^{-1}(B) \cap F^B) \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0).$$

Since $B = \bigcup \{ V \in \mathcal{B}_P : \overline{V} \subseteq B \}$, we can find a $V \in \mathcal{B}_P$ with $\overline{V} \subseteq B$ such that

$$h \upharpoonright (h^{-1}(V) \cap F^B) \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0).$$

In the end, define

$$E = \{ x \in F^B : \forall k (x \in U_k \Rightarrow h \upharpoonright (h^{-1}(V) \cap U_k) \notin \operatorname{dec}(\boldsymbol{\Sigma}_1^0, \boldsymbol{\Delta}_3^0)) \}.$$

Then we have $E \cap D \neq \emptyset$, and

$$h \upharpoonright (h^{-1}(V) \cap U \cap E) \notin \operatorname{dec}(\mathbf{\Sigma}_1^0, \mathbf{\Delta}_3^0)$$

for any open set with $U \cap E \neq \emptyset$. Note that $\overline{V}^c \supseteq B^c$. So V, E and F^B satisfy clauses (a)–(c) as desired.

In the rest of this section, we fix X be a Polish space, Y a separable metrizable space, and $f: X \to Y$ a Σ_3^0 -measurable function.

Definition 4.2. Let $\mathcal{F} = \langle F_0, \cdots, F_k \rangle$ be a finite sequence of closed sets of X with $F_0 \supseteq \cdots \supseteq F_k$, U an open subset of X, and let $\mathcal{P} = \langle P_0, \cdots, P_k \rangle$ be a sequence of pairwise disjoint subsets of Y.

(i) If k = 0, i.e., $\mathcal{F} = \langle F_0 \rangle$, $\mathcal{P} = \langle P_0 \rangle$, then we say \mathcal{F} is \mathcal{P} -sharp in U if $U \cap F_0 \neq \emptyset$, and for any open set $U' \subseteq U$ with $U' \cap F_0 \neq \emptyset$, we have

$$f \upharpoonright (f^{-1}(P_0) \cap F_0 \cap U^r) \notin \operatorname{dec}(\Sigma_1^\circ, \Delta_3^\circ).$$

We also say F_0 itself is P_0 -sharp in U for brevity.

(ii) If k > 0, then we say \mathcal{F} is \mathcal{P} -sharp in U if F_k is P_k -sharp in U, and for any open set $U' \subseteq U$ with $U' \cap F_k$, $\mathcal{F} \upharpoonright k$ is $\mathcal{P} \upharpoonright k$ -sharp in some open set $U'' \subseteq U'$.

Proposition 4.3. Suppose $\mathcal{F} = \langle F_0, \cdots, F_k \rangle$ is \mathcal{P} -sharp in U. Then for any $U' \subseteq U$ with $U' \cap F_k \neq \emptyset$, we have \mathcal{F} is \mathcal{P} -sharp in U'.

Proposition 4.4. Suppose \mathcal{F} is \mathcal{P} -sharp in U. Then for any $m < \mathrm{lh}(\mathcal{F})$, $\mathcal{F} \upharpoonright m$ is $\mathcal{P} \upharpoonright m$ -sharp in some open set $U' \subseteq U$.

Let $\mathcal{P} = \langle P_0, \cdots, P_k \rangle$, $0 \leq j \leq l$, and let $C \subseteq P_j$. We denote

 $\mathcal{P} \setminus C = \langle P_0, \cdots, P_j \setminus C, \cdots, P_k \rangle.$

Lemma 4.5. Let $\mathcal{F} = \langle F_0, \dots, F_k \rangle$, $\mathcal{P} = \langle P_0, \dots, P_k \rangle$. Suppose \mathcal{F} is P-sharp in U. Let $0 \leq j \leq k$ and $(C_l)_{l < m}$ be a sequence of pairwise disjoint closed subsets of P_j . Then there exist at most one l such that \mathcal{F} is not $\mathcal{P} \setminus C_l$ -sharp in any open set $U' \subseteq U$.

Proof. We begin with k = j = 0. Without loss of generality, suppose there exists an l < m, say, l = 0, such that F_0 is not $P_0 \setminus C_0$ -sharp in U. Then there exists an open set $U_0 \subseteq U$ with $U_0 \cap F_0 \neq \emptyset$ such that

$$f \upharpoonright (f^{-1}(P_0 \setminus C_0) \cap F_0 \cap U_0) \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0).$$

Assume for contradiction that there exists $l \neq 0$ such that F_0 is not $P_0 \setminus C_l$ sharp in U_0 , then there is an open set $U_l \subseteq U_0$ with $U_l \cap F_0 \neq \emptyset$ such that

 $f \upharpoonright (f^{-1}(P_0 \setminus C_l) \cap F_0 \cap U_l) \in \operatorname{dec}(\boldsymbol{\Sigma}_1^0, \boldsymbol{\Delta}_3^0).$

Since C_0 and C_l are disjoint closed subsets of P_0 , Proposition 2.1 gives

 $f \upharpoonright (f^{-1}(P_0) \cap F_0 \cap U_l) \in \operatorname{dec}(\boldsymbol{\Sigma}_1^0, \boldsymbol{\Delta}_3^0),$

contradicting that F_0 is P_0 -sharp in U.

For k > 0, assume that we have proved for all k' < k.

Case 1. If j = k, since F_k is P_k -sharp in U, from the arguments for k = 0above, we may assume that there is an open set $U_0 \subseteq U$ with $U_0 \cap F_k \neq \emptyset$ such that F_k is $P_k \setminus C_l$ -sharp in U_0 for any $l \neq 0$. It follows that \mathcal{F} is $\mathcal{P} \setminus C_l$ -sharp U_0 for any $l \neq 0$.

Case 2. If j < k, assume for contradiction that there are more than one l, say, l = 0, 1, such that \mathcal{F} is not $\mathcal{P} \setminus C_l$ -sharp in any open set $U' \subseteq U$. Particularly, \mathcal{F} is not $\mathcal{P} \setminus C_0$ -sharp in U. Note that F_k is $P_k \setminus C_l$ -sharp in U for any l < m, so there exists an $U_0 \subseteq U$ with $U_0 \cap F_k \neq \emptyset$ such that $\mathcal{F} \upharpoonright k$ is not $(\mathcal{P} \upharpoonright k) \setminus C_0$ -sharp in any open set $U' \subseteq U_0$. Similarly, we can find an open set $U_1 \subseteq U_0$ with $U_1 \cap F_k \neq \emptyset$ such that $\mathcal{F} \upharpoonright k$ is not $(\mathcal{P} \upharpoonright k) \setminus C_1$ -sharp in any $U' \subseteq U_1$. By Propositions 4.3 and 4.4, there is an open set $U^* \subseteq U_1$ such that $\mathcal{F} \upharpoonright k$ is \mathcal{P} -sharp in U^* , contradicting the induction hypothesis. \square

Theorem 4.6. Let X be a Polish space, Y a separable metrizable space, and let $f: X \to Y$ be of Baire class 1. If $f^{-1}\Sigma_3^0 \subseteq \Sigma_3^0$, then $f \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$.

Proof. Assume for contradiction that $f \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$. We will define a continuous embedding $\psi : 2^{\omega} \to X$ and an G_{δ} set $G \subseteq Y$ such that $\psi^{-1}(f^{-1}(Y \setminus G)) = \Omega$. Thus $f^{-1}(G)$ is Π^0_3 -complete subset of X, contradicting $f^{-1}\Sigma_3^0 \subseteq \Sigma_3^0$.

It it well known that Y is homeomorphic to a subspace of \mathbb{R}^{ω} . Without loss of generality, we may assume $Y = \mathbb{R}^{\omega}$. Granting this assumption, we can fix a sequence of continuous functions $f_n: X \to Y$ pointwisely converging to f. Fix a compatible metric d on X with $d \leq 1$.

For $s \neq \emptyset$, let $\ln(s) = \lceil i, j \rceil + 1$. Now we **redefine** inheritors and innovators. We say s is an **inheritor** if j > 0 and $s(\lceil k, i+j-k\rceil) = 0$ for any $k \le i$ (note: it was for any k < i in the definition of inheritor in Theorem 3.6), otherwise we say s is an **innovator**. Note that s is always an innovator if j = 0 or $s(\lceil i, j \rceil) = 1$.

We will inductively construct for each $s \in 2^{<\omega}$ an open set V_s of Y, two closed sets E_s, F_s of X, an open set U_s of X, and a sequence of open sets $(U_s^w)_{s \prec w \prec s^{\frown} 0}$ of X satisfying the following:

- $(0) \operatorname{diam}(\overline{U_s}) \le 2^{-\mathrm{lh}(s)}, U_{s \cap 0} \cap U_{s \cap 1} = \emptyset, \overline{U_{s \cap 0}}, \overline{U_{s \cap 1}} \subseteq U_s^w;$
- (1) $F_{s^{\uparrow}1} \subseteq F_{s^{\uparrow}0};$ (2) $\overline{V}_s \subseteq V_{\emptyset}$ and $F_s \subseteq E_{\emptyset}$ for any $s \neq \emptyset;$
- (3) for any $s, t \neq \emptyset$ with row(s) = row(t), we have $V_s = V_t$ or $\overline{V_s} \cap \overline{V_t} = \emptyset$;
- (4) if $\operatorname{col}(s) > 0$, then $\overline{V_{s \cap 0}}, \overline{V_{s \cap 1}} \subseteq V_s$ and $F_{s \cap 0} \subseteq F_s$;
- (5) $E_s \subseteq F_s;$
- (6) $E_s \cap U_s^w \neq \emptyset$ for each w;
- (7) $U_s = U_s^s$, and $U_s^{w_1} \supseteq U_s^{w_2}$ for $w_1 \preceq w_2$;
- (8) if s is an inheritor, then we have

$$V_s = V_{v(s)}, \quad F_s = F_{v(s)}, \quad E_s = E_{u(s)};$$

(9) if s is an innovator, then $\overline{V_s} \cap \overline{V_t} = \emptyset$ for any t with $t \prec s$ and row(t) =row(s); furthermore, there exists $n \ge \ln(s)$ such that $f_n(U_s) \subseteq V_s$;

(10) if
$$s \neq \emptyset$$
, by letting $V_s^- = \begin{cases} V_{s \restriction (\ln(s)-1)}, & \operatorname{row}(s) > 0, \\ V_{\emptyset}, & \operatorname{row}(s) = 0, \end{cases}$
$$P_s^r = V_s^- \setminus \bigcup \{ \overline{V_t} : t \leq r, \operatorname{row}(t) = \operatorname{row}(s) \},$$
$$\mathcal{P}_s^r = \langle P_{s \restriction (\ln(s) - \operatorname{row}(s))}^r, \cdots, P_s^r, V_s \rangle,$$
$$\mathcal{F}_s = \langle F_{s \restriction (\ln(s) - \operatorname{row}(s))}, \cdots, F_s, E_s \rangle,$$

then for any $t \leq s < t^{0}$, we have

- (a) if t^{0} is an innovator, then \mathcal{F}_{t} is \mathcal{P}_{t}^{s} -sharp in U_{t}^{s} ;
- (b) if t^{0} is an inheritor, then $\mathcal{F}_{u(t^{0})}$ is $\mathcal{P}_{u(t^{0})}^{s}$ -sharp in U_{t}^{s} .

When we complete the construction, for any $z \in 2^{\omega}$, we set $\psi(z)$ to be the unique element of $\bigcap_k U_{z \restriction k}$. From (0) and (7), ψ is continuous embedding from 2^{ω} to X. Put

$$G_m = \bigcup_{\operatorname{row}(t)=m} V_t, \quad G = \bigcap_{m < \omega} G_m.$$

If $z \in \Omega$, there exist $i_0 < \omega$ and a strictly increasing sequence $j_k > 0$ with $z(\lceil i_0, j_k \rceil) = 1$ for any $k < \omega$. Since $z \upharpoonright (\lceil i_0, j_k \rceil + 1)$ is an innovator, by (9), there is $n_k > \lceil i_0, j_k \rceil$ such that $f_{n_k}(\psi(z)) \in V_{z \upharpoonright (\lceil i_0, j_k \rceil + 1)}$. It follows from (9) that $f(\psi(z)) \notin V_t$ whenever row $(t) = i_0$. Thus

$$f(\psi(z)) \notin G_{i_0} \supseteq G.$$

If $z \notin \Omega$, we show that $f(\psi(z)) \in G$. For any $m < \omega$, there exists $J_m < \omega$ such that $z(\lceil i, j \rceil) = 0$ for any $i \leq m$ and any $j > J_m$. So $z \upharpoonright (\lceil m, j \rceil + 1)$ is an inheritor for any $j > J_m$. Denote

$$V_m = V_{z \upharpoonright (\ulcorner m, J_m \urcorner + 1)}, \quad u_j^m = u(z \upharpoonright (\ulcorner m, j \urcorner + 1)).$$

By (8), we have $V_m = V_{z | (\lceil m, j \rceil + 1)}$ for all $j > J_m$. Since all u_j^m are innovators, by (9) we can find an $n_j \ge \ln(u_j^m)$ such that $f_{n_j}(\psi(z)) \in V_{u_j^m}$. By (4) and (8) we have $f_{n_j}(\psi(z)) \in V_m$ for all $j > J_m$. So $f(\psi(z)) \in \overline{V_m}$ for each m. Again by (4) we have $\overline{V_{m+1}} \subseteq V_m$ for any $m < \omega$. So

$$f(\psi(z)) \in \overline{V_{m+1}} \subseteq V_m \subseteq G_n$$

for all $m < \omega$. It follows that $f(\psi(z)) \in G$.

Now we turn to the construction.

First, set D, P, h, \mathcal{B}_P as follows:

(i) P = Y, D = X, h = f;

(ii) \mathcal{B}_P is a countable basis of Y.

Applying Lemma 4.1 with these D, P, h, \mathcal{B}_P , we get an open set V of Y and two closed sets $E \subseteq F$ of X. Then put

$$V_{\emptyset} = V, \quad F_{\emptyset} = F, \quad E_{\emptyset} = E, \quad U_{\emptyset} = X.$$

Secondly, assume that we have constructed V_t, E_t, F_t, U_t , and U_t^w for $t, w \prec s \uparrow 0$. We will define for $s \uparrow 0$ and $s \uparrow 1$. We consider the following two cases:

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Case 1. Assume s^0 is an inheritor. Let $v = v(s^0), u = u(s^0)$. Put

$$V_{s^\frown 0} = V_v, \quad F_{s^\frown 0} = F_v, \quad E_{s^\frown 0} = E_u,$$

Note that either s = u, or s is also an inheritor with u(s) = u, so $E_s = E_u$. By (7), $E_u \cap U_s^{s_M} \neq \emptyset$, so we can define an open set $U_{s \cap 0}$ such that

$$\overline{U_{s^{\frown}0}} \subseteq U_s^{s_M}, \quad U_{s^{\frown}0} \cap E_u \neq \emptyset, \quad \operatorname{diam}(\overline{U_{s^{\frown}0}}) \le 2^{-(\operatorname{lh}(s)+1)}.$$

We set $U_{s \cap 0}^{s \cap 0} = U_{s \cap 0}$ and $U_t^{s \cap 0} = U_t^{s_M}$ for other t.

To check (0)–(10), the only nontrivial one is (10)(a) with $t = s^{\circ}0$. Note that, if $s^{\circ}0^{\circ}0$ is innovator, then $\operatorname{col}(s^{\circ}0^{\circ}0) = 0$, i.e., u = v, so $\mathcal{P}_{s^{\circ}0}^{s^{\circ}0} = \mathcal{P}_{u}^{s_{M}}$ and $\mathcal{F}_{s^{\circ}0} = \mathcal{F}_{u}$. Since (10) holds for s_{M} , it holds for $s^{\circ}0$ too.

By shrinking, we may assume $E_u \cap (U_s^{s_M} \setminus \overline{U_{s^{\frown}0}}) \neq \emptyset$. By (10)(b) and Proposition 4.3, we see \mathcal{F}_u is $\mathcal{P}_u^{s_M}$ -sharp in $U_s^{s_M} \setminus \overline{U_{s^{\frown}0}}$. By Proposition 4.4, $\mathcal{F}_u \upharpoonright (\operatorname{row}(v) + 1)$ is $\mathcal{P}_u^{s_M} \upharpoonright (\operatorname{row}(v) + 1)$ -sharp in some open set $U \subseteq (U_s^{s_M} \setminus \overline{U_{s^{\frown}0}})$. Thus F_v is $\mathcal{P}_v^{s_M}$ -sharp in U, and hence

$$f \upharpoonright (f^{-1}(P_s^{s_M}) \cap F_v \cap U) \notin \operatorname{dec}(\mathbf{\Sigma}_1^0, \mathbf{\Delta}_3^0).$$

We inductively define V^l, E^l , and F^l for each $l < \omega$. Denote $F^{-1} = F_v$. Assume that we have defined V^k, E^k , and F^k for k < l. Set D, P, h, \mathcal{B}_P as follows:

(i) $P = P_s^{s_M} \setminus \bigcup_{k < l} \overline{V^k}, D = F^{l-1} \cap U \cap f^{-1}(P), h = f \upharpoonright D;$

(ii) \mathcal{B}_P is a countable basis of P such that $\overline{V} \subseteq P$ for each $V \in \mathcal{B}_P$.

Applying Lemma 4.1 with these D, P, h, \mathcal{B}_P , we get an open set V of Y and two closed sets $E \subseteq F \subseteq \overline{D} \subseteq F^{l-1}$ with $E \cap U \supseteq E \cap D \neq \emptyset$. Denote $V^l = V, E^l = E$, and $F^l = F$. This complete the induction.

For $s \prec t \preceq s^{\circ}0$, if $t^{\circ}0$ is an innovator, it follows from (10)(a) that \mathcal{F}_t is $\mathcal{P}_t^{s^{\circ}0}$ -sharp in $U_t^{s^{\circ}0}$. By Lemma 4.5, we can find an natural number L_t such that, for any $l \ge L_t$, \mathcal{F}_t is $\mathcal{P}_t^{s^{\circ}0} \setminus \overline{V^l}$ -sharp in some $U_t^l \subseteq U_t^{s^{\circ}0}$. If $t^{\circ}0$ is an inheritor, from (10)(b) and Lemma 4.5, we can also find an natural number L_t such that, for any $l \ge L_t$, $\mathcal{F}_{u(t^{\circ}0)}$ is $\mathcal{P}_t^{s^{\circ}0} \setminus \overline{V^l}$ -sharp in some open set $U_t^l \subseteq U_t^{s_M}$. Then we set

$$L = \max\{L_t : s \prec t \preceq s^{\frown}0\}$$

and $U_t^{s^{\uparrow}1} = U_t^L$ for $t \leq s^{\uparrow}0 \prec t^{\uparrow}0$, i.e., for $s \prec t \leq s^{\uparrow}0$.

From Lemma 4.1 and $F^L \subseteq E_s$, we can see that $(\mathcal{F}_s \upharpoonright \operatorname{row}(s^{-1}))^{\frown} F^{L^{\frown}} E^L$ is $(\mathcal{P}_s^{s_M} \setminus \overline{V^L})^{\frown} V^L$ -sharp in U.

Pick an $x \in (f^{-1}(V^L) \cap E^L \cap U)$. Since $f(x) \in V^L$, there is an $n > \ln(s)$ such that $f_n(x) \in V^L$. Then we can define an open set $U_{s^{-1}}$ such that

$$\overline{U_{s^{\uparrow}1}} \subseteq U, \quad f_n(U_{s^{\uparrow}1}) \subseteq V^L, \quad x \in U_{s^{\uparrow}1}, \quad \operatorname{diam}(\overline{U_{s^{\uparrow}1}}) \le 2^{-(\operatorname{lh}(s)+1)}.$$

Then put

$$V_{s^{\frown}1} = V^L, \quad E_{s^{\frown}1} = E^L, \quad F_{s^{\frown}1} = F^L,$$

and $U_{s^{\frown}1}^{s^{\frown}1} = U_{s^{\frown}1}$.

Case 2. Assume $s \cap 0$ is an innovator. Since $s \leq s_M \prec s \cap 0$, by (10)(a), we have \mathcal{F}_s is $\mathcal{P}_s^{s_M}$ -sharp in $U_s^{s_M}$. Thus E_s is V_s -sharp in $U_s^{s_M}$, and hence

$$f \upharpoonright (f^{-1}(V_s) \cap E_s \cap U_s^{s_M}) \notin \operatorname{dec}(\boldsymbol{\Sigma}_1^0, \boldsymbol{\Delta}_3^0).$$

Set D, P, h, \mathcal{B}_P as follows:

(i)
$$P = V_s, D = E_s \cap U_s^{s_M} \cap f^{-1}(P), h = f \upharpoonright D;$$

(ii) \mathcal{B}_P is a countable basis of P such that $\overline{V} \subseteq P$ for each $V \in \mathcal{B}_P$.

Applying Lemma 4.1 with these D, P, h, \mathcal{B}_P , we get an open set V of Yand two closed sets $E \subseteq F \subseteq \overline{D} \subseteq E_s$ with $E \cap U_s^{s_M} \supseteq E \cap D \neq \emptyset$. From Lemma 4.1 and $F \subseteq E_s$, we can see that $(\mathcal{F}_s \upharpoonright \operatorname{row}(s^{-}0))^{-}F^{-}E$ is $(\mathcal{P}_s^{s_M} \setminus \overline{V})^{-}V$ -sharp in $U_s^{s_M}$.

Pick an $x \in (f^{-1}(V) \cap E \cap U_s^{s_M})$. Since $f(x) \in V$, there is an $n > \ln(s)$ such that $f_n(x) \in V$. Then we can an open set $U_{s \cap 0}$ such that

$$\overline{U_{s^{\frown}0}} \subseteq U_s^{s_M}, \quad f_n(U_{s^{\frown}0}) \subseteq V, \quad x \in U_{s^{\frown}0}, \quad \operatorname{diam}(\overline{U_{s^{\frown}0}}) \le 2^{-(\operatorname{lh}(s)+1)}.$$

Then put

$$V_{s^\frown 0}=V,\quad E_{s^\frown 0}=E,\quad F_{s^\frown 0}=F,$$

and $U_{s^{\frown}0}^{s^{\frown}0} = U_{s^{\frown}0}$, and $U_t^{s^{\frown}0} = U_t^{s_M}$ for other t.

To check (0)–(10), it is trivial for $s = \emptyset$. For $s \neq \emptyset$, the only nontrivial clauses are (3), (9), and (10). Note that $\operatorname{row}(s^{\circ}0) > 0$, so $V_{s^{\circ}0}^{-} = V_s$. Note also that either s is also an innovator, or $\operatorname{col}(s^{\circ}0) = 0$, i.e., u(s) = v(s). In both cases, (4) and (9) imply that there is no $t \prec s^{\circ}0$ such that $\operatorname{row}(t) =$ $\operatorname{row}(s^{\circ}0)$ and $V_t^{-} = V_s$. So (3) and (9) hold. Therefore, $P_{s^{\circ}0}^{s^{\circ}0} = V_s \setminus \overline{V}$, thus $\mathcal{P}_{s^{\circ}0}^{s^{\circ}0} = (\mathcal{P}_s^{s_M} \setminus \overline{V})^{\circ}V$. Similarly, $\mathcal{P}_t^{s^{\circ}0} = \mathcal{P}_t^{s_M}$ and $\mathcal{P}_{u(t^{\circ}0)}^{s^{\circ}0} = \mathcal{P}_{u(t^{\circ}0)}^{s_M}$ for $t \prec s^{\circ}0 \prec t^{\circ}0$. Since (10) holds for s_M , it holds for $s^{\circ}0$ too.

By shrinking, we may assume $F \cap (U_s^{s_M} \setminus \overline{U_{s \cap 0}}) \neq \emptyset$. By Lemma 4.1,

$$f \upharpoonright (f^{-1}(V_s \setminus \overline{V}) \cap F \cap (U_s^{s_M} \setminus \overline{U_{s \cap 0}})) \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0).$$

Now we define for s^1 similar to the way in Case 1.

Theorem 4.7. Let X be a Polish space, Y a separable metrizable space, and let $f: X \to Y$. Then

$$f^{-1}\Sigma_3^0 \subseteq \Sigma_3^0 \iff f \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0).$$

Proof. The " \Leftarrow " part is trivial, we only prove the " \Rightarrow " part.

Since $f^{-1}\Sigma_3^0 \subseteq \Sigma_3^0$ implies $f^{-1}\Sigma_2^0 \subseteq \Sigma_3^0$, it follows from Theorem 3.6 that $f \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$, i.e., there exists a sequence of G_{δ} set X_n such that $\bigcup_n X_n = X$ and each $f \upharpoonright X_n$ is of Baire class 1. Then Theorem 4.6 gives $f \upharpoonright X_n \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$. Therefore, we have $f \in \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$.

Corollary 4.8. Let X be a Polish space, Y a separable metrizable space, and let $f : X \to Y$. If $f \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$, then there exists a Cantor set $C \subseteq X$ such that $f \upharpoonright C \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$. *Proof.* If $f \notin \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$, by Corollary 3.7, there exists a Cantor set $C \subseteq X$ such that $f \upharpoonright C \notin \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$. It is clear that $f \upharpoonright C \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$.

If $f \in \operatorname{dec}(\Sigma_2^0, \Delta_3^0)$, i.e., there exists a sequence of G_{δ} set X_n such that $\bigcup_n X_n = X$ and each $f \upharpoonright X_n$ is of Baire class 1, then there is some X_n such that $f \upharpoonright X_n \notin \operatorname{dec}(\Sigma_1^0, \Delta_3^0)$. Let ψ be the continuous embedding defined in Theorem 4.6. Put $C = \psi(2^{\omega})$.

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(Longyun Ding and Jiafei Zhao) School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071, P.R.China

E-mail address: dinglongyun@gmail.com (Longyun Ding), 294465868@qq.com (Jiafei Zhao)

(TAKAYUKI KIHARA) GRADUATE SCHOOL OF INFORMATICS, NAGOYA UNIVERSITY, NAGOYA, 464-8601, JAPAN

E-mail address: kihara@i.nagoya-u.ac.jp

(BRIAN SEMMES) STUDYLAB LANGUAGE SCHOOL, MOSCOW, NIKOLSKAYA ST. 10, NIKOLSKAYA PLAZA OFFICE CENTRE, RUSSIAN FEDERATION

E-mail address: brian@studylab.ru