# ON EFFECTIVELY CLOSED SETS OF EFFECTIVE STRONG MEASURE ZERO

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Abstract. The strong measure zero sets of reals have been widely studied in the context of set theory of the real line. The notion of strong measure zero is straightforwardly effectivized. A set of reals is said to be of *effective strong measure zero* if for any computable sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive rationals, a sequence of intervals $I_n$ of diameter $\varepsilon_n$ covers the set. We observe that a set is of effective strong measure zero if and only if it is of measure zero with respect to any outer measure constructed by Monroe's Method from a computable atomless outer premeasure defined on all open balls. This measure-theoretic restatement permits many characterizations of strong measure zero in terms of semimeasures as well as martingales. We show that for closed subsets of Cantor space, effective strong nullness is equivalent to another well-studied notion called *diminutiveness*, the property of not having a computably perfect subset. Further, we prove that if $P$ is a nonempty effective strong measure zero $\Pi^0_1$ set consisting only of noncomputable elements, then some Martin-Löf random reals computes no element in $P$, and $P$ has an element that computes no autocomplex real. Finally, we construct two different special $\Pi^0_1$ sets, one of which is not of effective strong measure zero, but consists only of infinitely-often K-trivial reals, and the other is perfect and of effective strong measure zero, but contains no anti-complex reals.

## 1. Introduction

1.1. **Background.** Miniaturization of set-theoretic notions is sometimes useful in computability theory. For example, set-theoretic forcing is transformed into a notion called arithmetical forcing and $n$-generic reals, which has become a fundamental tool in computability theory. There is another set-theoretical notion whose miniaturization we expect to play an important role. The notion is known as *strong measure zero* which was introduced by Émile Borel in 1919. Careful consideration of the measure theoretic behavior of sets of reals has profound significance in the study of algorithmic randomness [14, 30]. Binns [5, 6, 7] conducted a deep study of notions stronger than being of measure zero/Hausdorff dimension zero, and clarified an interesting

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connection among such measure theoretic smallness, Muchnik degrees, and Kolmogorov complexity.

In his thesis in 2011, Kihara pointed out the relationship between Binns' smallness properties [5, 6, 7] and the notion of small sets in set theory of the real line [9]. Kihara introduced the notion of *effective strong measure zero* to formalize his idea. In Section 2, we will see that a set of reals is of effective strong measure zero if and only if for any computable atomless[1] outer measure defined on all open balls, the set is of measure zero with respect to the outer measure. This characterization urges us to study other effectivizations of strong measure zero. As one such effectivization, we study *strong Martin-Löf measure zero* introduced in a personal communication between Kihara and Miyabe in 2012. A set of reals is called strong Martin-Löf measure zero if for any computable atomless outer premeasure defined on all open balls, the set contains no Martin-Löf random real with respect to the outer measure induced by the premeasure.

It is known that the notion of Martin-Löf randomness (nullness, and Martin-Löf nullness) admits many natural characterizations such as incompressibility (in terms of Kolmogorov complexity) and unpredictability (in terms of martingales). In Section 2, we will focus on characterizations of Martin-Löf randomness by semimeasures, Kolmogorov complexity, and martingales, and extend such characterizations to Martin-Löf nullness with respect to any outer measure induced by a computable outer premeasure. This leads to the conclusion that the concept of effective strong measure zero is robust enough to have many characterizations just as in the case of Martin-Löf reals.

In Section 3, we review the results of Higuchi/Kihara [21] in their research on the $\Pi^0_1$ sets of reals of effective strong measure zero as well as their Muchnik degrees. In contrast to Laver's model [27] of $\mathsf{ZFC}$ in which all strongly measure zero sets are countable, one can easily construct an effectively strongly measure zero set of reals that is uncountable and $\Pi^0_1$ definable. Indeed, the class of uncountable $\Pi^0_1$ definable effective strong measure zero subsets of Cantor space has nontrivial properties. We see that for closed sets of reals, effective strong measure zero is equivalent to another well-studied notion called *diminutiveness* [7], the property of not having a computably perfect subset. Further, we prove that if $P$ is a nonempty effective strong measure zero $\Pi^0_1$ set consisting only of noncomputable elements, then some Martin-Löf random real computes no element in $P$, and $P$ has an element that computes no autocomplex real. Here, an infinite binary sequence $x$ is (*auto-*)*complex* if there exists an ($x$-)computable function $f$ such that $\mathrm{K}(x \upharpoonright f(n)) \geq n$ for all $n \in \mathbb{N}$, where $K$ denotes the prefix-free Kolmogorov complexity.

[1] A point which has a positive $\mu$-measure is called an *atom* of $\mu$. A measure having an atom is called *atomic*. Otherwise, it is called *atomless*. As pointed out by L. A. Levin in 1970, every computable real can be $\mu$-random for a computable *atomic* probability measure $\mu$. We avoid such a singular case by restricting the range of $\mu$ to atomless measures.

In Section 4, we see some interactions between measure theoretic smallness and Kolmogorov complexity. We prove two non-basis theorems for small $\Pi^0_1$ sets and very small $\Pi^0_1$ sets. By using the non-basis results, we construct a computably perfect $\Pi^0_1$ set consisting only of non-generic reals that are both complex and infinitely often K-trivial, and we also construct a perfect (but effectively strongly measure zero) $\Pi^0_1$ set consisting only of non-generic reals that are neither complex nor anti-complex. Here, an infinite binary sequence $x \in 2^{\mathbb{N}}$ is *infinitely often* K*-trivial* if there exists a constant $c$ such that $\mathrm{K}(x \upharpoonright n) \leq \mathrm{K}(n) + c$ for infinitely many $n \in \mathbb{N}$, and $x$ is *anti-complex* if there exists an ($x$-)computable function $f$ such that $\mathrm{K}(x \upharpoonright f(n)) \leq n$ for all $n \in \mathbb{N}$.

1.2. **Notation.** Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ denote the set of all natural numbers; $\mathbb{N}^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \mathbb{N}\}$, Baire space; $2^{\mathbb{N}} = \{f \mid f : \mathbb{N} \to \{0, 1\}\}$, Cantor space; $\mathbb{N}^{<\mathbb{N}}$, the set of all finite strings of natural numbers; and $2^{<\mathbb{N}}$, the set of all finite binary strings. We define $\mathbb{N}^{\leq\mathbb{N}} = \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$ and $2^{\leq\mathbb{N}} = 2^{<\mathbb{N}} \cup 2^{\mathbb{N}}$. We use $\emptyset$ to denote the empty string or the empty set. For a set $A$, we use $\#A$ to denote the cardinal number of $A$. For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ and $\rho, \rho' \in \mathbb{N}^{\leq\mathbb{N}}$, we use $\sigma \subset \rho$ to mean that $\sigma$ is an initial segment of $\rho$, i.e., $\rho$ extends $\sigma$; $\sigma \mid \tau$ to mean that $\sigma$ and $\tau$ are incomparable, i.e., neither $\sigma \subset \tau$ nor $\sigma \supset \tau$; $\sigma\rho$ or $\sigma^\frown\rho$ to denote the concatenation of $\sigma$ and $\rho$, i.e., the string $\sigma$ followed by $\rho$; $|\rho|$ and $\mathrm{lh}(\rho)$ to denote the length of $\rho$, i.e., the cardinal number of the domain of $\rho$; $\rho \upharpoonright n$ to denote the initial segment of $\rho$ of the length $n$ for any $n \leq |\rho|$; $\rho \cap \rho'$ to denote the longest common initial segment of $\rho$ and $\rho'$; $\rho \oplus \rho'$ to denote the string $\rho''$ with $\rho''(2n) = \rho(n)$ and $\rho''(2n+1) = \rho'(n)$, when $|\rho| = |\rho'|$ or $|\rho| = |\rho'|+1$; $[\![\sigma]\!]$ to denote the set $\{f \in \mathbb{N}^{\mathbb{N}} : \sigma \subset f\}$ or the set $\{f \in 2^{\mathbb{N}} : \sigma \subset f\}$ depending on the context. For $n \in \mathbb{N}$, $\{0, 1\}^n$ denotes the set of all binary strings of length $n$; $\{0, 1\}^{\leq n}$, the set of all binary strings of length $\leq n$. We often identify a natural number $n$ with the string $\langle n \rangle$ of the length 1. Let $A \subset \mathbb{N}^{<\mathbb{N}}$ and $P, Q \subset \mathbb{N}^{\mathbb{N}}$. $A$ is *prefix-free* if $\sigma \mid \tau$ for any distinct two element $\sigma, \tau \in A$. $[\![A]\!]$ denotes the set $\bigcup_{\sigma \in A}[\![\sigma]\!]$; $[A]$, the set $\{f \in \mathbb{N}^{\mathbb{N}} : (\forall n \in \mathbb{N})[f \upharpoonright n \in A]\}$; $\mathrm{Ext}(P)$, the set $\{\sigma : [\![\sigma]\!] \cap P \neq \emptyset\}$; $\mathrm{Br}(P)$, the set $\{\sigma \cap \tau : \sigma, \tau \in \mathrm{Ext}(P)\ \&\ \sigma \mid \tau\}$; $\mathrm{Brl}(P)$, the set $\{|\sigma| : \sigma \in \mathrm{Br}(P)\}$; $\mathrm{Ext}(A)$, $\mathrm{Br}(A)$ and $\mathrm{Brl}(A)$ denote the sets $\mathrm{Ext}([A])$, $\mathrm{Br}([A])$ and $\mathrm{Brl}([A])$, respectively; $P \times Q$ denotes the set $\{f \oplus g : f \in P\ \&\ g \in Q\}$; and $P + Q$, the set $0P \cup 1Q$, where $0P = \{0f : f \in P\}$ and $1Q = \{1g : g \in Q\}$. A set $T \subset \mathbb{N}^{<\mathbb{N}}$ is called a *tree* if $T$ is closed under taking initial segments, i.e., $\tau \in T$ if $\tau \subset \sigma$ for some $\sigma \in T$. For a tree $T$, $\sigma \in T$ is an *immediate successor* of $\tau$ in $T$ if $\tau \subset \sigma$ and $|\sigma| = |\tau| + 1$; $T$ is *finitely branching* if every element in $T$ has at most finitely many immediate successors.

We always treat Baire space $\mathbb{N}^{\mathbb{N}}$ and Cantor space $2^{\mathbb{N}}$ as topological spaces whose open sets are of the form $[\![A]\!]$ for some subset $A$ of $\mathbb{N}^{<\mathbb{N}}$ or $2^{<\mathbb{N}}$. An open set $U$ of $\mathbb{N}^{\mathbb{N}}$ or $2^{\mathbb{N}}$ is *c.e.* or $\Sigma^0_1$ if there exists a c.e. set $A$ with $U = [\![A]\!]$. The complement of a c.e. open set is called *co-c.e. closed* or $\Pi^0_1$. The $\Pi^0_1$ sets are characterized as the sets of the form $[T]$ for some computable tree $T$.

Let $X$ be $\mathbb{N}$ or $2^{<\mathbb{N}}$. A function $G : X \to \mathbb{R}$ is *computable* if there exists a computable function $g : \mathbb{N} \times X \to \mathbb{Q}$ such that $|G(a) - g(n, a)| < n^{-1}$ for any $a \in X$ and $n \in \mathbb{N}$. In addition, if $g(n, a) \leq G(a)$ for any $a \in X$ and $n \in \mathbb{N}$, then $G$ is *left-c.e.*, and if $g(n, a) \geq G(a)$ for any $a \in X$ and $n \in \mathbb{N}$, then $G$ is *right-c.e.*

Let $P, Q \subset \mathbb{N}^{\mathbb{N}}$. $P$ is *Medvedev reducible* (or *strongly reducible*) to $Q$, denoted by $P \leq_{\mathrm{s}} Q$, if there is a computable function $\Phi : Q \to P$; $P$ is *Medvedev comparable* with $Q$ if $P \leq_{\mathrm{s}} Q$ or $P \geq_{\mathrm{s}} Q$; otherwise, *Medvedev incomparable*; $P$ is *Medvedev equivalent* to $Q$, denoted by $P \equiv_{\mathrm{s}} Q$, if $P \leq_{\mathrm{s}} Q$ and $P \geq_{\mathrm{s}} Q$. The *Medvedev degree* of $P$ is the equivalence class of $P$ under the equivalence relation $\equiv_{\mathrm{s}}$. $P$ is *Muchnik reducible* (or *weakly reducible*) to $Q$, denoted by $P \leq_{\mathrm{w}} Q$, if $P \leq_{\mathrm{s}} \{g\}$ for all $g \in Q$. *Muchnik comparability*, *Muchnik incomparability*, *Muchnik equivalence* and *Muchnik degree* are defined in the same way. The arithmetical hierarchy is introduced in the usual way. We refer the reader to several textbooks [14, 30, 36] to know some basic terminologies and facts of Computability Theory.

## 2. Effective Strong Measure Zero

In this section we give definitions of two main concepts, effective strong measure zero and strong Martin-Löf measure zero, which we discuss throughout the paper. It is known that the notion of Martin-Löf randomness has many characterizations in terms of Kolmogorov complexity, semimeasures, and martingales. We will extend the characterization results to generalized Martin-Löf randomness with respect to an arbitrary outer measure, and then it turns out that the concepts of effective strong measure zero and strong Martin-Löf measure zero are robust enough to have a lot of characterizations.

### 2.1. Outer Measures.

Émil Borel in 1919 introduced the notion of strong measure zero. A subset $X$ of a metric space is *strong measure zero* (or *strong null*) if for any sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers, there exists a sequence $\{I_n\}_{n\in\mathbb{N}}$ of open intervals such that $X \subset \bigcup_{n\in\mathbb{N}} I_n$ and $\mathrm{diameter}(I_n) \leq 2^{-k_n}$ for all $n \in \mathbb{N}$. This notion is straightforwardly effectivized in the following manner.

**Definition 1** (Kihara)**.** A subset $X$ of $2^{\mathbb{N}}$ is said to be of *effective strong measure zero* (or *effective strong null*) if for any computable sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers, there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of finite binary strings such that $X \subset \bigcup_{n\in\mathbb{N}} [\![\sigma_n]\!]$ and $|\sigma_n| \geq k_n$ for all $n \in \mathbb{N}$.

This notion can be characterized as a measure-theoretic concept.

**Definition 2.** A function $\mu$ from $2^{<\mathbb{N}}$ into $[0, \infty)$. is *monotone* if $\mu(\sigma) \geq \mu(\sigma i)$ for any $\sigma \in 2^{<\mathbb{N}}$ and $i \in \{0, 1\}$. It is *subadditive* if $\mu(\sigma) \leq \mu(\sigma 0) + \mu(\sigma 1)$ holds for any $\sigma \in 2^{<\mathbb{N}}$. It is called *atomless* if $\liminf_{n\to\infty} \mu(f \restriction n) = 0$ for any $f \in 2^{\mathbb{N}}$. An *outer premeasure* is a monotone subadditive atomless function.

Our definition of outer premeasures is essentially equivalent to the notion of *premeasures defined on the (cl)open subsets of* $2^{\mathbb{N}}$ in the sense of Rogers [33]. Every outer premeasure is naturally extended to an outer measure by a so-called "*Method I construction*" (named by Munroe; see Rogers [33]).

**Definition 3.** For a monotone function $\mu : 2^{<\mathbb{N}} \to [0, \infty)$, we define a function $\mu^*$ from the power set of $2^{\mathbb{N}}$ into $[0, \infty]$ by

$$\mu^*(X) = \inf \left\{ \sum_{\sigma \in A} \mu(\sigma) : A \subset 2^{<\mathbb{N}}, \text{ and } X \subset [\![A]\!] \right\}.$$

We call the function $\mu^*$ the *induced outer measure* by $\mu$. A subset $X$ of $2^{\mathbb{N}}$ is said to be *$\mu$-null* or of *$\mu$-zero* if $\mu^*(X) = 0$.

Note that an outer premeasure $\mu$ is atomless if and only if the induced outer measure $\mu^*$ is atomless, i.e., $\mu^*(\{x\}) = 0$ for every single point $x \in 2^{\mathbb{N}}$. Moreover, given premeasure $\mu : 2^{<\mathbb{N}} \to [0, \infty)$, one can effectively obtain a probability premeasure (i.e., $\tilde{\mu} : 2^{<\mathbb{N}} \to [0, 1]$) such that the classes of all $\mu$-null reals and all $\tilde{\mu}$-null reals coincide. Hereafter, we only consider probability premeasures.

Of course, there are several other methods to construct a measure from a premeasure. For instance, the notion of Hausdorff $h$-measure $\mathcal{H}^h$ is obtained from a so-called "Method II construction" (named by Munroe; see Rogers [33]). However, the concept of "$\mathcal{H}^h$-nullness" is also obtained as the "$\mu_h$-nullness" by taking $\mu_h(\sigma) = h(2^{-|\sigma|})$ for every binary string $\sigma$ (see also Reimann [31]).

**Theorem 4.** *A subset $X$ of $2^{\mathbb{N}}$ is of effective strong measure zero if and only if $X$ is of $\mu$-zero for all atomless computable outer premeasures $\mu : 2^{<\mathbb{N}} \to [0, 1]$.*

*Proof.* First, suppose that $X$ is of effective strong measure zero. Fix an atomless computable outer premeasure $\mu : 2^{<\mathbb{N}} \to [0, 1]$. By the compactness of $2^{\mathbb{N}}$, there exists a computable strictly increasing function $F : \mathbb{N} \to \mathbb{N}$ such that $\mu(\sigma) < 2^{-n}$ holds for any $n \in \mathbb{N}$ and $\sigma \in \{0, 1\}^{F(n)}$. Define $\mu' : 2^{<\mathbb{N}} \to [0, 1]$ by $\mu'(\sigma) = 2^{-n_\sigma}$ for all $\sigma \in 2^{<\mathbb{N}}$ and $n_\sigma = \min\{n \in \mathbb{N} : |\sigma| < F(n+1)\}$. It is easy to see that $\mu'$ is an atomless computable outer premeasure and $\mu(\sigma) \leq \mu'(\sigma)$ for any $\sigma \in 2^{<\mathbb{N}}$. Take an arbitrary natural number $m$. Since $X$ is of effective strong measure zero, there exists a sequence $\{\sigma_n\}_{n \in \mathbb{N}}$ of finite binary strings such that $X \subset \bigcup_{n \in \mathbb{N}} [\![\sigma_n]\!]$ and $|\sigma_n| \geq F(n + m)$. We have the following inequality

$$\mu^*(X) \leq (\mu')^*(X) \leq \sum_{n \in \mathbb{N}} \mu'(\sigma_n) \leq \sum_{n \in \mathbb{N}} 2^{-(n+m+1)} = 2^{-m}.$$

Since we take $m$ arbitrarily, $X$ is of $\mu$-zero.

Second, suppose that $X$ is of $\mu$-zero for all atomless computable outer premeasures $\mu : 2^{<\mathbb{N}} \to [0, 1]$. To show that $X$ is of effective strong measure zero, fix a computable sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers. Choose a computable strictly increasing function $F:\mathbb{N}\to\mathbb{N}$ such that $F(n)\geq\max\{k_m : m<2^n\}$ for all $n\in\mathbb{N}$. Define $\mu:2^{<\mathbb{N}}\to[0,1]$ by $\mu(\sigma)=2^{-n_\sigma}$ for all $\sigma\in 2^{<\mathbb{N}}$, where $n_\sigma=\min\{n\in\mathbb{N}:|\sigma|<F(n+1)\}$. Obviously, $\mu$ is an atomless computable outer premeasure. Since $X$ is of $\mu$-zero, there exists a subset $A$ of $2^{<\mathbb{N}}$ such that $X\subset[\![A]\!]$ and $\sum_{\sigma\in A}\mu(\sigma)<1$. Choose an initial segment $N$ of $\mathbb{N}$ and a sequence $\{\sigma_n\}_{n\in N}$ of finite binary strings such that $n\mapsto\sigma_n$ is a bijection from $N$ onto $A$ with $|\sigma_{n-1}|\leq|\sigma_n|$ for all $n\in N$. We show that $|\sigma_n|\geq k_n$ for any $n\in N$. Fix $n\in N$. Choose the maximum number $n_0\in N$ such that $2^{n_0}-1\leq n$. If $|\sigma_n|<k_n$, then $|\sigma_m|\leq|\sigma_n|<k_n\leq F(n_0+1)$ for any $m<2^{n_0}$ and, therefore, $\sum_{m<2^{n_0}}\mu(\sigma_m)\geq\sum_{m<2^{n_0}}2^{-n_0}\geq 1$. Since $\sum_{n\in N}\mu(\sigma_n)<1$, we have $|\sigma_n|\geq k_n$. Thus $X$ is of effective strong measure zero. □

**Remark 5.** Omitting "effective" and "computable" from the proof of Theorem 4, we have a proof of the theorem obtained by omitting the same words from Theorem 4. This characterization of strong measure zero of $2^{\mathbb{N}}$ is a counterpart of a characterization of $\mathbb{R}$ proved by Besicovitch [3, Theorem 1] in 1933. Thus, Theorem 4 can be seen as an effective version of Besicovitch's theorem.

It may be natural to study concepts obtained from the latter condition of the precede theorem by strengthening effectivity. The next definition gives one of such concepts.

**Definition 6.** For an outer premeasure $\mu:2^{<\mathbb{N}}\to[0,1]$, a subset $X$ of $2^{\mathbb{N}}$ is called *Martin-Löf $\mu$-null* or of *Martin-Löf $\mu$-zero* if there exists a computable descending sequence $\{U_n\}_{n\in\mathbb{N}}$ of c.e. open subsets of $2^{\mathbb{N}}$ such that $X\subset\bigcap_{n\in\mathbb{N}}U_n$ and $\mu^*(U_n)\leq 2^{-n}$ for any $n\in\mathbb{N}$.

**Definition 7** (Kihara/Miyabe)**.** A subset $X$ of $2^{\mathbb{N}}$ is of *strong Martin-Löf measure zero* (or *strongly Martin-Löf null*) if for any atomless computable outer premeasure $\mu:2^{<\mathbb{N}}\to[0,1]$, $X$ is of Martin-Löf $\mu$-zero.

**Remark 8.** A set of reals is *universally null* or *universal measure zero* if it is null with respect to all Borel atomless probability measures. See, for instance, Bukovský [9, Chapter 8]. Clearly, every strong measure zero set of reals is universally measure zero. There can be at least three nontrivial concepts of *effective universal measure zero.* The first effectivization of the notion of universal measure zero was studied by van Lambalgen [37], where he said that a set of reals is *constructively small* if it is null with respect to all computable atomless probability measures. The second concept is introduced by Bienvenu/Porter [4]. A real is contained in $\mathsf{NCR}_{\mathrm{comp}}$ if it is not Martin-Löf random with respect to all computable atomless probability measures. Further, as the third concept, a real is *never continuously random* [32, 1] if it is not Martin-Löf random with respect to all Borel atomless probability measures.

**Remark 9.** According to [20, Definition 2.2, Definition 8.1], for a function $\mu : 2^{<\mathbb{N}} \to [0,1]$ and $A \subset 2^{<\mathbb{N}}$, the *direct $\mu$-weight* of $A$, the *prefix-free $\mu$-weight* of $A$, the *vehement $\mu$-weight* of $A$ are defined as $\mathrm{dwt}_\mu(A) = \sum_{\sigma\in A}\mu(\sigma)$, $\mathrm{pwt}_\mu(A) = \sup\{\mathrm{dwt}_\mu(P) : P \subset A \text{ is prefix-free}\}$, and $\mathrm{vwt}_\mu(A) = \inf\{\mathrm{dwt}_\mu(S) : [\![A]\!] \subset [\![S]\!]\}$, respectively. In Definition 6, $\mu^*(U_n)$ corresponds to the vehement weight. Thus, Definition 6 has two other variants. However our main results on strong Martin-Löf measure zero do not depend on the choice of these three definitions.

Even when we replace "computable outer premeasure" with "right-c.e. outer premeasure" in Theorem 4, the theorem still holds. Similarly, strong Martin-Löf measure zero is equivalent to the one obtained by replacing "computable outer premeasure" with "right-c.e. outer premeasure" in Definition 7. On the other hand, the same holds even when we replace "computable outer premeasure" with "exactly computable rational-valued outer premeasure". Here, a non-negative rational valued function $\mu : 2^{<\mathbb{N}} \to \mathbb{Q}_{\geq 0}$ is *exactly computable* if there exists a computable function $(f,g) : X \to \mathbb{N} \times (\mathbb{N} \setminus \{0\})$ such that $\mu(\sigma) = f(\sigma)/g(\sigma)$ for any $\sigma \in 2^{<\mathbb{N}}$. These facts are easy corollaries of the following lemmas. For a function $F$ from $2^{<\mathbb{N}}$ into a set, $F$ is called *length-preserving* if $F(\sigma) = F(\tau)$ for any $\sigma, \tau \in 2^{<\mathbb{N}}$ with $|\sigma| = |\tau|$.

**Lemma 10.** *For any right-c.e. atomless monotone function $\mu_0 : 2^{<\mathbb{N}} \to [0,1]$, we can find a computable atomless length-preserving outer premeasure $\mu_1 : 2^{<\mathbb{N}} \to (0,1]$ such that $\mu_0(\sigma) \leq \mu_1(\sigma)$ for any $\sigma \in 2^{<\mathbb{N}}$.*

*Proof.* By the compactness of $2^{\mathbb{N}}$, we can find a computable function $F : \mathbb{N} \to \mathbb{N}$ such that $F(n) < F(n+1)$ and $\mu_0(\sigma) < 2^{-n}$ for any $\sigma \in \{0,1\}^{F(n)}$. Define $\mu_1 : 2^{<\mathbb{N}} \to (0,1]$ by $\mu_1(\sigma) = 2^{-n_\sigma}$, where $n_\sigma = \min\{n \in \mathbb{N} : |\sigma| < F(n+1)\}$. It is easy to see that $\mu_1$ satisfies our desired properties. □

**Lemma 11.** *For any computable outer premeasure $\mu_0 : 2^{<\mathbb{N}} \to (0,1]$, we can find an exactly computable outer premeasure $\mu_1 : 2^{<\mathbb{N}} \to (0,1] \cap \mathbb{Q}$ such that $2^{-1}\mu_0(\sigma) \leq \mu_1(\sigma) \leq \mu_0(\sigma)$ for any $\sigma \in 2^{<\mathbb{N}}$. Therefore, $\mu_0$-zero and $\mu_1$-zero coincide and also Martin-Löf $\mu_0$-zero and Martin-Löf $\mu_1$-zero coincide.*

*Proof.* Fix a computable outer premeasure $\mu_0 : 2^{<\mathbb{N}} \to (0,1]$. By the properties of $\mu_0$, we can find an exactly computable function $\mu_1 : 2^{<\mathbb{N}} \to (0.1]$ such that $2^{-1}\mu_0(\sigma) < \mu_1(\sigma) < \mu_0(\sigma)$ and $\mu_1(\sigma i) \leq \mu_1(\sigma) \leq \mu_1(\sigma 0) + \mu_1(\sigma 1)$ for any $\sigma \in 2^{<\mathbb{N}}$ and $i \in \{0,1\}$. Clearly, $\mu_1$ is an outer premeasure with our desired properties. Note that for any subset $A \subset 2^{<\mathbb{N}}$, the inequalities

$$2^{-1}\sum_{\sigma\in A}\mu_0(\sigma) < \sum_{\sigma\in A}\mu_1(\sigma) < \sum_{\sigma\in A}\mu_0(\sigma)$$

holds. Thus for any $X \subset 2^{\mathbb{N}}$, we have $2^{-1}\mu_0^*(X) \leq \mu_1^*(X) \leq \mu_0^*(X)$. This is the reason why $\mu_0$-zero and $\mu_1$-zero coincide and so do Martin-Löf $\mu_0$-zero and Martin-Löf $\mu_1$-zero. □

Technically, it is important to show that for a given outer premeasure $\mu$ and a c.e. open set $U \subseteq 2^\omega$, one can effectively approximate the value of $\mu^*(U)$ in the following manner.

**Lemma 12** (see also Miller [29, Lemma 3.3])**.** *Let $\mu : 2^{<\mathbb{N}} \to [0,1] \cap \mathbb{Q}$ be an exactly computable outer premeasure, and let $A$ be a nonempty c.e. subset of $2^{<\mathbb{N}}$. We can uniformly find a c.e. subset $B$ of $2^{<\mathbb{N}}$ such that $[\![A]\!] \subset [\![B]\!]$ and*

$$\mu^*([\![A]\!]) = \mu^*([\![B]\!]) = \sup\left\{\sum_{\sigma \in B'} \mu(\sigma) : B' \text{ is a finite prefix-free subset of } B\right\}.$$

Note that if $A$ is finite, one can compute the value $\mu^*([\![A]\!])$ in the following way. For an outer premeasure $\mu : 2^{<\mathbb{N}} \to [0,1]$, we have $\mu(\sigma) = \mu^*([\![\sigma]\!])$ for any $\sigma \in 2^{<\mathbb{N}}$, and, moreover, for any finite set $A \subset 2^{<\mathbb{N}}$ there exists a finite set $B \subset 2^{<\mathbb{N}}$ such that $\mu^*([\![A]\!]) = \sum_{\sigma \in B} \mu(\sigma)$ and every $\sigma$ in $B$ has some extension in $A$ by subadditivity. This implies that for a computable outer premeasure $\mu$ and a c.e. open set $U \subset 2^{\mathbb{N}}$, $\mu^*(U)$ is left-c.e. uniformly in indices of $\mu$ and $U$. In the case that $\mu$ is an exactly computable rational-valued outer premeasure, the above $B$ can be computed uniformly in an index of $\mu$ and $A$. Therefore, $\mu^*([\![A]\!])$ can be computed uniformly.

*Proof of Lemma 12.* Let $F : \mathbb{N} \to A$ be a computable function onto $A$. We define $B = \bigcup_{s \in \mathbb{N}} B_s$ recursively as follows: Fix $s \in \mathbb{N}$. Suppose that we have constructed $B_t$ for any $t < s$. Choose the shortest $\tau \subset F(s)$ such that $\mu^*([\![\{F(s)\} \cup \bigcup_{t<s} B_t]\!]) = \mu^*([\![\{\tau\} \cup \bigcup_{t<s} B_t]\!])$. Define $B_s = \{\tau\} \cup \bigcup_{t<s} B_t$.

Let $A_s = \{F(t) : t \leq s\}$ and $C_s = \{\sigma \in B_s : (\forall \tau \subsetneq \sigma)[\tau \notin B_s]\}$ for any $s \in \mathbb{N}$. By induction on $s \in \mathbb{N}$, it is easy to see that $[\![A_s]\!] \subset [\![B_s]\!] = [\![C_s]\!]$, $\mu^*([\![A_s]\!]) = \mu^*([\![B_s]\!]) = \sum_{\sigma \in C_s} \mu(\sigma)$ and for any finite prefix-free subset $D$ of $B_s$, $\sum_{\sigma \in D} \mu(\sigma) \leq \sum_{\sigma \in C_s} \mu(\sigma)$ hold. Thus $[\![A]\!] \subset [\![B]\!]$ and $\mu^*([\![A]\!]) \geq \sum_{\sigma \in D} \mu(\sigma)$ hold for any finite prefix-free subset $D$ of $B$. We need to show that for any $n \in \mathbb{N}$ there exists a finite prefix-free subset $D_n$ of $B$ such that $\mu^*([\![B]\!]) - n^{-1} \leq \sum_{\sigma \in D_n} \mu(\sigma)$. Fix $n \in \mathbb{N}$. Choose any prefix-free subset $D$ of $B$ with $[\![D]\!] = [\![B]\!]$. Let $D_n$ be a finite subset of $D$ such that $(\sum_{\sigma \in D} \mu(\sigma)) - n^{-1} \leq \sum_{\sigma \in D_n} \mu(\sigma)$. □

2.2. **Semimeasures.** In 1973, L. A. Levin [28] gave a characterization of Martin-Löf $\mu$-randomness for an arbitrary computable probability measure $\mu$ on $2^{\mathbb{N}}$ by using the notion of semimeasure. Namely, an infinite binary sequence $x$ is Martin-Löf $\mu$-random if and only if the supremum of the ratios of the a priori probability of $[\![x \upharpoonright n]\!]$ to the $\mu$-probability of $[\![x \upharpoonright n]\!]$ is bounded. In this subsection, we generalize Levin's theorem for arbitrary outer measure, constructed by Method I, from a computable outer premeasure defined on all clopen sets, and we characterize effective strong measure zero and strong Martin-Löf measure zero in terms of semimeasure.

**Definition 13.** A function $\nu : 2^{<\mathbb{N}} \to [0,\infty)$ is called a *semimeasure* if $\nu(\sigma) \geq \nu(\sigma 0) + \nu(\sigma 1)$ holds for any $\sigma \in 2^{<\mathbb{N}}$. A left-c.e. semimeasure

$\nu : 2^{<\mathbb{N}} \to [0,\infty)$ is called *optimal* if for any left-c.e. semimeasure $\nu' : 2^{<\mathbb{N}} \to [0,\infty)$, there exists a natural number $c$ such that $\nu'(\sigma) \leq c\nu(\sigma)$ for any $\sigma \in 2^{<\mathbb{N}}$.

By the definition, every semimeasure is necessarily monotone. Levin found the following fact.

**Theorem 14** (Levin). *There exists an optimal left-c.e. semimeasure $\nu_{\rm opt} : 2^{<\mathbb{N}} \to [0,1]$.*

Levin [28] showed that for every computable probability measure $\mu$ on $2^{\mathbb{N}}$, a real $x \in 2^{\mathbb{N}}$ is not Martin-Löf $\mu$-random if and only if we have

$$\limsup_{n\to\infty} \frac{\nu_{\rm opt}(x \upharpoonright n)}{\mu(x \upharpoonright n)} = \infty.$$

In this case, we say that *the ratio of $\nu_{\rm opt}$ to $\mu$ is unbounded at a point $x$*. The following theorem generalizes Levin's theorem to an arbitrary outer measure constructed by Method I, from a computable outer premeasure defined on all clopen sets.

**Theorem 15** (essentially, Higuchi/Hudelson/Simpson/Yokoyama [20, Theorem 2.8])**.** *Let $\mu : 2^{<\mathbb{N}} \to (0,1]$ be a computable outer premeasure. A subset $X$ of $2^{\mathbb{N}}$ is of Martin-Löf $\mu$-zero if and only if the ratio of $\nu_{\rm opt}$ to $\mu$ is unbounded at any $x \in X$.*

*Proof.* We follow the argument of [20, Theorem 2.8]. By Lemma 11, we may assume that $\mu$ is positive rational valued, and exactly computable.

First, suppose that $X$ is of Martin-Löf $\mu$-zero. Choose a computable descending sequence $\{U_n\}_{n\in\mathbb{N}}$ of c.e. open sets such that $X \subset \bigcap_{n\in\mathbb{N}} U_n$ and $\mu^*(U_n) \leq 2^{-n}$ for any $n \in \mathbb{N}$. By Lemma 12, there is a computable sequence $\{B_n\}_{n\in\mathbb{N}}$ of c.e. subsets of $2^{<\mathbb{N}}$ such that $U_n \subset [\![B_n]\!]$ and $\mu^*(U_n) = \mu^*([\![B_n]\!]) = \sup\{\sum_{\sigma\in B'} \mu(\sigma) : B' \text{ is a finite prefix-free subset of } B_n\}$ for any $n \in \mathbb{N}$. Let $B_n^\sigma = \{\tau \in B_n : \tau \supset \sigma\}$ for any $\sigma \in 2^{<\mathbb{N}}$ and $n \in \mathbb{N}$. For a natural number $n$, define $\nu_n : 2^{<\mathbb{N}} \to [0,1]$ by

$$\nu_n(\sigma) = \sup\left\{\sum_{\tau\in B'} \mu(\tau) : B' \text{ is a finite prefix-free subset of } B_n^\sigma\right\}.$$

Then $\nu_n$ is left-c.e., uniformly in $n \in \mathbb{N}$. For any finite prefix-free $M \subset B_n^{\sigma 0}$ and any finite prefix-free $N \subset B_n^{\sigma 1}$, $M \cup N$ is a finite prefix-free subset of $B_n^\sigma$, and, therefore,

$$\nu_n(\sigma) \geq \sum_{\tau\in M\cup N} \mu(\tau) = \sum_{\tau\in M} \mu(\tau) + \sum_{\tau\in N} \mu(\tau)$$

holds. Thus $\nu_n(\sigma) \geq \nu_n(\sigma 0) + \nu_n(\sigma 1)$ holds for all $\sigma \in 2^{<\mathbb{N}}$. In other words, $\nu_n$ is semimeasure. Define $\nu : 2^{<\mathbb{N}} \to [0,1]$ by $\nu(\sigma) = \sum_{n\in\mathbb{N}} \nu_{2n}(\sigma)2^{n-1}$. Here, note that $\nu_{2n}(\sigma) \leq \nu_{2n}(\emptyset) = \mu^*(U_{2n}) \leq 2^{-2n}$. Hence, $\nu(\sigma) \leq 1$.

Then, $\nu$ is a semimeasure since so is $\nu_{2n}$ for any $n \in \mathbb{N}$, and $\nu$ is left-c.e., since $\nu_{2n}$ is left-c.e. uniformly in $n$. If $\sigma \in B_{2n}$, then $2^{n-1}\mu(\sigma) = 2^{n-1}\nu_{2n}(\sigma) \leq \nu(\sigma)$. Thus $\sup_{\sigma \subsetneq f} \nu(\sigma)/\mu(\sigma) = \infty$ for any $f \in X$ since $X \subset \bigcap_{n\in\mathbb{N}} U_n \subset \bigcap_{n\in\mathbb{N}} [\![B_n]\!]$.

Second, suppose that there exists a left-c.e. semimeasure $\nu : 2^{<\mathbb{N}} \to [0,1]$ such that $\sup_{\sigma \subsetneq f} \nu(\sigma)/\mu(\sigma) = \infty$ holds for any $f \in X$. For a natural number $n$, let $A_n = \{\sigma \in 2^{<\mathbb{N}} : \nu(\sigma) > \mu(\sigma)2^n\}$. Since $\nu$ is left-c.e. and $\mu$ is computable, $A_n$ is c.e. uniformly in $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have $[\![A_n]\!] \supset [\![A_{n+1}]\!] \supset X$, and $\sigma \in A_n$ implies $\mu(\sigma) < \nu(\sigma)2^{-n}$. Define $B_n = \{\sigma \in A_n : (\forall \tau \subsetneq \sigma)[\tau \notin A_n]\}$ for each $n \in \mathbb{N}$. Since any two distinct element in $B_n$ are incomparable and $\nu$ is a semimeasure, the inequation

$$\mu^*([\![A_n]\!]) \leq \sum_{\sigma \in B_n} \mu(\sigma) < \sum_{\sigma \in B_n} \nu(\sigma)2^{-n} \leq \nu(\emptyset)2^{-n} \leq 2^{-n}$$

holds. Thus $X$ is of Martin-Löf $\mu$-zero via $\{[\![A_n]\!]\}_{n\in\mathbb{N}}$. □

**Corollary 16.** *A subset $X$ of $2^{\mathbb{N}}$ is of strong Martin-Löf measure zero if and only if for any computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$, the ratio of $\nu_{\mathrm{opt}}$ to $\mu$ is unbounded at any point $x \in X$.*

Omitting "computable", "left-c.e." and "Martin-Löf" appeared in the proof of Theorem 15, we have a proof of the following corollary.

**Corollary 17.** *Let $\mu : 2^{<\mathbb{N}} \to (0,1]$ be an outer premeasure. A subset $X$ of $2^{\mathbb{N}}$ is of $\mu$-zero if and only if there is a semimeasure $\nu : 2^{<\mathbb{N}} \to [0,1]$ such that the ratio of $\nu$ to $\mu$ is unbounded at any $x \in X$.*

By Theorem 4, Remark 5, Corollary 17, and the proof of Corollary 16, we have another characterizations of effective strong measure zero and strong measure zero.

**Corollary 18.** *A subset $X$ of $2^{\mathbb{N}}$ is of (effective) strong measure zero if and only if, for any (computable) atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$, there exists a semimeasure $\nu : 2^{<\mathbb{N}} \to [0,1]$ such that the ratio of $\nu$ to $\mu$ is unbounded at any $x \in X$.*

2.3. **Kolmogorov complexity.** The main theorem in algorithmic randomness theory is that the notion of Martin-Löf randomness is characterized as "incompressibility" in the sense of Kolmogorov complexity. In this subsection, we characterize the notion of strong Martin-Löf measure zero by using the notion of Kolmogorov complexity and a priori complexity. Hereafter, $\mathrm{K} : 2^{<\mathbb{N}} \to \mathbb{N}$ denotes a prefix-free Kolmogorov complexity.

**Definition 19** (Kjos-Hanssen/Merkle/Stephan [26], Kanovich [23, 24])**.** An infinite binary string $f \in 2^{\mathbb{N}}$ is called *complex* if there exists a computable unbounded increasing function $F : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{K}(\sigma) \geq F(|\sigma|)$ for any $\sigma \subsetneq f$.

**Definition 20** (Levin [28])**.** We define a right-c.e. function $\mathrm{KA} : 2^{<\mathbb{N}} \to [0,\infty)$ by $\mathrm{KA}(\sigma) = -\log_2 \nu_{\mathrm{opt}}(\sigma)$. We call KA *a priori complexity* or *a priori entropy.*

Actually, we can replace prefix-free Kolmogorov complexity K with a priori complexity KA to define "complex". In other words, $f \in 2^{\mathbb{N}}$ is complex if and only if there exists a computable unbounded increasing function $F : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{KA}(\sigma) \geq F(|\sigma|)$ for any $\sigma \subsetneq f$. See, for instance, [20, Remark 7.2].

**Theorem 21** (Kihara/Miyabe)**.** *A subset $X$ of $2^{\mathbb{N}}$ is of strong Martin-Löf measure zero if and only if $X$ includes no complex element.*

*Proof.* By Corollary 16, we know that $X$ is of strong Martin-Löf measure zero if and only if $\sup_{\sigma\subsetneq f} \nu_{\mathrm{opt}}(\sigma)/\mu(\sigma) = \infty$ for any $f \in X$ and any computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$. Since the function $\log_2$ is strictly increasing, it is equivalent to that $\sup_{\sigma\subsetneq f}(-\mathrm{KA}(\sigma) - \log_2 \mu(\sigma))$ diverges to infinity for any $f \in X$ and any computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$. By Lemma 10, it is equivalent to that $\sup_{\sigma\subsetneq f}(-\mathrm{KA}(\sigma) + F(|\sigma|)$ diverges to infinity for any $f \in X$ and any computable increasing unbounded function $F : \mathbb{N} \to [0,\infty)$. Here, we can clearly replace $F : \mathbb{N} \to [0,\infty)$ with $F : \mathbb{N} \to \mathbb{N}$. As a result, we now know that $X$ is of strong Martin-Löf measure zero if and only if

$$\sup_{\sigma\subsetneq f}(-\mathrm{KA}(\sigma) + F(|\sigma|) = \infty \tag{1}$$

holds for any $f \in X$ and any computable increasing unbounded function $F : \mathbb{N} \to \mathbb{N}$.

First, suppose that $X$ is of strong Martin-Löf measure zero. To see that $X$ includes no complex element, fix $f \in X$ and a computable unbounded increasing function $F : \mathbb{N} \to \mathbb{N}$. We show that $\mathrm{KA}(\sigma) < F(|\sigma|)$ for some $\sigma \subsetneq f$. By (1), there exists a finite binary string $\sigma \subsetneq f$ such that $-\mathrm{KA}(\sigma) + F(|\sigma|) > 0$. Thus we have $\mathrm{KA}(\sigma) < F(|\sigma|)$.

Second, suppose that $X$ is not of strong Martin-Löf measure zero. Choose a computable increasing unbounded function $F : \mathbb{N} \to \mathbb{N}$ which fails to satisfy (1). Choose $f \in X$ and $c \in \mathbb{N}$ such that $-\mathrm{KA}(\sigma) + F(|\sigma|) < c$ holds for any $\sigma \subsetneq f$. We have $\mathrm{KA}(\sigma) \geq F(|\sigma|) - c$ for any $\sigma \subsetneq f$. Hence the function $(n \mapsto \max\{F(n) - c, 0\})$ witnesses that $f$ is complex.

□

2.4. **Martingales.** As is well known, in 1930s, Jean Ville introduced the notion of martingale to characterize the property of measure zero. More precisely, the property of measure zero with respect to any generalized Bernoulli measure generated by a sequence of *biased* coins is characterized by using a generalized betting process based on a sequence of *odds* in terms of martingale. These characterizations will be straightforwardly generalized to all outer measures on Cantor space, constructed by Method I, from a computable outer premeasure defined on all clopen sets, by introducing the notion of *odds-function*. In the rest of this section, we introduce special kinds of martingales, and characterize effective strong measure zero and strong Martin-Löf measure zero in terms of martingales.

**Definition 22.** Any function $O : 2^{<\mathbb{N}} \to [1, \infty)$ is said to be *odds*. Let $M$ be a function from $2^{<\mathbb{N}}$ into $[0, \infty)$. The function $M$ is called a $O$*-martingale* if

$$M(\sigma) = \frac{M(\sigma 0)}{O(\sigma 0)} + \frac{M(\sigma 1)}{O(\sigma 1)}$$

holds for any $\sigma \in 2^{<\mathbb{N}}$. The function $M'$ is called a $O$*-supermartingale* if

$$M'(\sigma) \geq \frac{M'(\sigma 0)}{O(\sigma 0)} + \frac{M'(\sigma 1)}{O(\sigma 1)}$$

holds for any $\sigma \in 2^{<\mathbb{N}}$.

When $O$ is the constant function 2, i.e., $O(\sigma) = 2$ for any $\sigma \in 2^{<\mathbb{N}}$, then $O$-martingales are martingales, and $O$-supermartingales are supermartingales in the usual sense. (Here, $M : 2^{<\mathbb{N}} \to [0, \infty)$ is a *martingale* if $2M(\sigma) = M(\sigma 0) + M(\sigma 1)$ for any $\sigma \in 2^{<\mathbb{N}}$. Similarly, $M' : 2^{<\mathbb{N}} \to [0, \infty)$ is a *supermartingale* if $2M'(\sigma) \geq M'(\sigma 0) + M'(\sigma 1)$ for any $\sigma \in 2^{<\mathbb{N}}$.)

Intuitively, a $O$-martingale $M$ is a strategy of a gambler for the following game: at stage $s$ the gambler has a history $\sigma \in \{0, 1\}^s$ of the game and a capital $M(\sigma)$. The gambler should divide $M(\sigma)$ into two $M(\sigma 0)/O(\sigma 0)$, $M(\sigma 1)/O(\sigma 1)$ to bet on the next $\{0, 1\}$-value. The capital at stage $s + 1$ is $M(\sigma 0) = O(\sigma 0) \cdot M(\sigma 0)/O(\sigma 0)$ if the value is 0, and $M(\sigma 1)$ otherwise. Of course, the history at stage $s + 1$ is $\sigma i$ if the value is $i \in \{0, 1\}$.

**Definition 23.** For odds $O : 2^{<\mathbb{N}} \to [1, \infty)$ and an $O$-supermartingale $M : 2^{<\mathbb{N}} \to [0, \infty)$, define $\mu_O : 2^{<\mathbb{N}} \to (0, 1]$ and $\nu^O_M : 2^{<\mathbb{N}} \to [0, \infty)$ by $\mu_O(\sigma) = (\prod_{\tau \subset \sigma} O(\tau))^{-1}$ and $\nu^O_M(\sigma) = M(\sigma)\mu_O(\sigma)$. Conversely, for a monotone function $\mu : 2^{<\mathbb{N}} \to (0, 1]$ and a semimeasure $\nu : 2^{<\mathbb{N}} \to [0, \infty)$, define $O_\mu : 2^{<\mathbb{N}} \to [1, \infty)$ by

$$O_\mu(\emptyset) = \frac{1}{\mu(\emptyset)}, \qquad O_\mu(\sigma i) = \frac{\mu(\sigma)}{\mu(\sigma i)}$$

and define $M^\mu_\nu : 2^{<\mathbb{N}} \to [0, \infty)$ by $M^\mu_\nu(\sigma) = \nu(\sigma)/\mu(\sigma)$.

To state the next proposition, let us denote $\mathcal{F}$ and $\mathcal{G}$ for the functions $(O, M) \mapsto (\mu_O, \nu^O_M)$ and $(\mu, \nu) \mapsto (O_\mu, M^\mu_\nu)$, respectively.

**Proposition 24.** *$\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are identity functions.*

**Lemma 25.** *Let $O : 2^{<\mathbb{N}} \to [1, \infty)$ be odds and let $M : 2^{<\mathbb{N}} \to [0, \infty)$ be an $O$-supermartingale. Then $\mu_O : 2^{<\mathbb{N}} \to (0, 1]$ is a monotone function and $\nu^O_M : 2^{<\mathbb{N}} \to [0, \infty)$ is a semimeasure. Moreover, $\mu_O$ is an outer premeasure provided $O(\sigma 0)^{-1} + O(\sigma 1)^{-1} \geq 1$ holds for any $\sigma \in 2^{<\mathbb{N}}$.*

*Proof.* It is easy to see that $\mu_O$ is monotone. We know that $\nu_M^O$ is a semimeasure by the following inequalities:

$$\begin{aligned}\nu_M^O(\sigma 0) + \nu_M^O(\sigma 1) &= M(\sigma 0)\mu_O(\sigma 0) + M(\sigma 1)\mu_O(\sigma 1) \\ &= \left(\frac{M(\sigma 0)}{O(\sigma 0)} + \frac{M(\sigma 1)}{O(\sigma 1)}\right)\mu_O(\sigma) \leq M(\sigma)\mu_O(\sigma) = \nu_M^O(\sigma).\end{aligned}$$

Suppose that the odds $O$ satisfies $O(\sigma 0)^{-1} + O(\sigma 1)^{-1} \geq 1$ for any $\sigma \in 2^{<\mathbb{N}}$. Using this assumption and the definition of $\mu_O$, we have the following inequalities: $\mu_O(\sigma 0)+\mu_O(\sigma 1) = \mu_O(\sigma)(O(\sigma 0)^{-1}+O(\sigma 1)^{-1}) \geq \mu_O(\sigma)$. Thus $\mu_O$ is an outer premeasure. □

Conversely, we have the following theorem.

**Lemma 26.** *Let $\mu : 2^{<\mathbb{N}} \to (0,1]$ be a monotone function and let $\nu : 2^{<\mathbb{N}} \to [0,\infty)$ be an semimeasure. Then $O_\mu : 2^{<\mathbb{N}} \to [1,\infty)$ is odds and $M_\nu^\mu : 2^{<\mathbb{N}} \to [0,\infty)$ is an $O_\mu$-supermartingale. If $\mu$ is an outer premeasure, then $O_\mu(\sigma 0)^{-1} + O_\mu(\sigma 1)^{-1} \geq 1$ holds for any $\sigma \in 2^{<\mathbb{N}}$.*

*Proof.* It is clear that $O_\mu$ is odds. We know that $M_\nu^\mu$ is an $O_\mu$-supermartingale by the following inequalities:

$$\begin{aligned}\frac{M_\nu^\mu(\sigma 0)}{O_\mu(\sigma 0)} + \frac{M_\nu^\mu(\sigma 1)}{O_\mu(\sigma 1)} &= \frac{\nu(\sigma 0)}{\mu(\sigma 0)} \cdot \frac{\mu(\sigma 0)}{\mu(\sigma)} + \frac{\nu(\sigma 1)}{\mu(\sigma 1)} \cdot \frac{\mu(\sigma 1)}{\mu(\sigma)} \\ &= \frac{\nu(\sigma 0) + \nu(\sigma 1)}{\mu(\sigma)} \leq \frac{\nu(\sigma)}{\mu(\sigma)} = M_\nu^\mu(\sigma).\end{aligned}$$

Suppose that $\mu$ is an outer premeasure. We have the following inequalities: $O_\mu(\sigma 0)^{-1} + O_\mu(\sigma 1)^{-1} = \mu(\sigma)^{-1}(\mu(\sigma 0) + \mu(\sigma 1)) \geq \mu(\sigma)^{-1}\mu(\sigma) = 1$. □

**Definition 27.** Let $O$ be odds. An $O$-supermartingale $M$ *succeeds on* a subset $X$ of $2^{\mathbb{N}}$ if the capital $\sup_{\sigma \subsetneq f} M(\sigma)$ diverges to infinity for all $f \in X$.

Intuitively, if $M$ succeeds on $X$, then a gambler earns however he or she wants when the infinite $\{0,1\}$-sequence of the game is in $X$ and the gambler use $M$ as his or her strategy.

**Definition 28.** We say that odds $O$ is *fair* if $O(\sigma 0)^{-1} + O(\sigma 1)^{-1} = 1$ holds for each $\sigma \in 2^{<\mathbb{N}}$.

**Remark 29.** By the same argument from Lemma 25 and 26, it is easy to see that $\mu_O$ is a measure whenever $O$ is fair, and that $O_\mu$ is fair whenever $\mu$ is a measure. Moreover, for fair odds $O$, if an $O$-martingale $M$ succeeds on $X \subseteq 2^{\mathbb{N}}$, then there is also an $O$-martingale $M'$ such that $\lim_{n\to\infty} M(f \upharpoonright n)$ diverges to infinity for all $f \in X$. Here, if an outer premeasure $\mu$ satisfies $\mu(\sigma) = \mu(\sigma 0) + \mu(\sigma 1)$, then it is called a *measure*.

**Definition 30.** Odds $O : 2^{<\mathbb{N}} \to [1,\infty)$ is said to be *acceptable* if the value $\prod_{\sigma \subsetneq f} O(\sigma)$ diverges to infinity for any $f \in 2^{\mathbb{N}}$.

Intuitively, If odds $O$ is acceptable, then a gambler can earn however he or she wants when he or she bets properly for any infinite binary sequence.

Note that odds $O$ is acceptable if and only if $\mu_O$ is atomless by the definition of $\mu_O$. Also, a monotone function $\mu$ is atomless if and only if $O_\mu$ is acceptable. Now we give our characterizations of effective strong measure zero and strong Martin-Löf zero in terms of martingales.

**Theorem 31.** *A subset $X$ of $2^{\mathbb{N}}$ is of effective strong measure zero if and only if, for any computable acceptable odds $O : 2^{<\mathbb{N}} \to [1, \infty)$, there exists an $O$-supermartingale $M : 2^{<\mathbb{N}} \to [0, \infty)$ such that $M$ succeeds on $X$.*

*Proof.* Recall from Corollary 18 that the property of effective strong measure zero is characterized in terms of semimeasure.

First, suppose that $X$ satisfies the semimeasure condition in Corollary 18. Fix computable acceptable odds $O : 2^{<\mathbb{N}} \to [1, \infty)$. Define $O' : 2^{<\mathbb{N}} \to [1, 2]$ by $O'(\sigma) = \max\{O(\sigma), 2\}$. Then $O'$ is computable acceptable odds. Moreover, $O'(\sigma 0)^{-1} + O'(\sigma 1)^{-1} \geq 1$ for any $\sigma \in 2^{<\mathbb{N}}$. Thus, by Lemma 25, $\mu = \mu_{O'} : 2^{<\mathbb{N}} \to (0, 1]$ is a computable atomless outer premeasure. By our assumption, there exists a semimeasure $\nu : 2^{<\mathbb{N}} \to [0, 1]$ such that $\sup_{\sigma \subsetneq f} \nu(\sigma)/\mu(\sigma)$ diverges to infinity for all $f \in X$. By Lemma 26, $M = M^\mu_\nu$ is an $O'$-supermartingale since Proposition 24 implies $O_{\mu_{O'}} = O'$. Moreover, we can see that $M$ succeeds on $X$ since $M(\sigma) = \nu(\sigma)/\mu(\sigma)$ for any $\sigma \in 2^{<\mathbb{N}}$. By the definition of $O'$, we have $O'(\sigma) \leq O(\sigma)$ for any $\sigma \in 2^{<\mathbb{N}}$. Thus

$$\frac{M(\sigma 0)}{O(\sigma 0)} + \frac{M(\sigma 1)}{O(\sigma 1)} \leq \frac{M(\sigma 0)}{O'(\sigma 0)} + \frac{M(\sigma 1)}{O'(\sigma 1)} \leq M(\sigma)$$

for any $\sigma \in 2^{<\mathbb{N}}$. This implies that $M$ is $O$-supermartingale.

We next show the converse direction. Suppose that $X$ satisfies the martingale condition, and we will show that $X$ satisfies the semimeasure condition in Corollary 18. Fix a computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0, 1]$. By our assumption, choose an $O_\mu$-supermartingale $M : 2^{<\mathbb{N}} \to [0, \infty)$ such that $M$ succeed on $X$. Choose $n \in \mathbb{N}$ such that $n > M(\emptyset)$. Define an $O_\mu$-supermartingale $M'$ by $M'(\sigma) = M(\sigma)/n$. Note that $M'(\emptyset) \leq 1$ and $M'$ also succeeds on $X$. Put $\nu = \nu^{O_\mu}_{M'}$. Then $\nu : 2^{<\mathbb{N}} \to [0, \infty)$ is a semimeasure by Lemma 25, and we can also see $\sup_{\sigma \subsetneq f} \nu(\sigma)/\mu(\sigma) = \infty$ for all $f \in X$, since we have $M'(\sigma) = \nu(\sigma)/\mu(\sigma)$ by the following equality

$$\nu(\sigma) = \nu^{O_\mu}_{M'} = M'(\sigma)\mu_{O_\mu}(\sigma) = M'\mu(\sigma)$$

for any $\sigma \in 2^{<\mathbb{N}}$, where the last equation follows from Proposition 24. Moreover, $\nu(\sigma) \leq 1$ for any $\sigma \in 2^{<\mathbb{N}}$ since $\nu$ is monotone and $\nu(\emptyset) = M'(\emptyset)/\mu(\emptyset) \leq 1$. Thus we have the desired condition. $\square$

Note that $M^\mu_\nu$ is left-c.e. if $\mu$ is computable and $\nu$ is left-c.e., and $\nu^O_M$ is left-c.e. if $O$ is computable and $M$ is left-c.e. Hence, we have the same characterization for strong Martin-Löf zero.

2.5. **Characterizations.** The characterization results in this section is summarized as follows.

**Corollary 32.** *For a subset $X$ of $2^{\mathbb{N}}$, the following are pairwise equivalent:*

(1) *$X$ is of (effective) strong measure zero.*
(2) *$X$ is of $\mu$-zero for any (computable) atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to [0,1]$.*
(3) *Every (computable) atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$ has a semimeasure $\nu : 2^{<\mathbb{N}} \to [0,1]$ such that the ratio of $\nu$ to $\mu$ is unbounded at any $x \in X$.*
(4) *For any (computable) acceptable odds $O : 2^{<\mathbb{N}} \to [1,\infty)$, there exists an $O$-supermartingale $M : 2^{<\mathbb{N}} \to [0,\infty)$ such that $M$ succeeds on $X$.*

Recall that $\nu_{\mathrm{opt}} : 2^{<\mathbb{N}} \to [0,1]$ denotes a fixed optimal left-c.e. semimeasure.

**Corollary 33.** *For a subset $X$ of $2^{\mathbb{N}}$, the following are pairwise equivalent:*

(1) *$X$ is of strong Martin-Löf measure zero.*
(2) *$X$ contains no complex element.*
(3) *The ratio of $\nu_{\mathrm{opt}}$ to $\mu$ is unbounded at any $x \in X$ for any computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$.*
(4) *For any computable acceptable odds $O : 2^{<\mathbb{N}} \to [1,\infty)$, there exists a left-c.e. $O$-supermartingale $M : 2^{<\mathbb{N}} \to [0,\infty)$ such that $M$ succeeds on $X$.*

Note that by applying Corollary 33 to a singleton, we can obtain the similar characterizations of complex strings in terms of semimeasures and martingales. We also have the similar results for *universal measure zero sets* (see Remark 8). For instance, we have the following results.

**Corollary 34.** *For a subset $X$ of $2^{\mathbb{N}}$, the following are pairwise equivalent:*

(1) *$X$ is of universal measure zero.*
(2) *For any atomless measure $\mu : 2^{<\mathbb{N}} \to (0,1]$, there exists a semimeasure $\nu : 2^{<\mathbb{N}} \to [0,1]$ such that the ratio of $\mu$ and $\nu$ is unbounded (equivalently, $\liminf_{n\to\infty} \nu(x \upharpoonright n)/\mu(x \upharpoonright n)$ diverges to infinity) at any $x \in X$.*
(3) *For any acceptable fair odds $O$, there exists an $O$-supermartingale $M : 2^{<\mathbb{N}} \to [0,\infty)$ such that $M$ succeeds on $X$.*

**Corollary 35.** *For $x \in 2^{\mathbb{N}}$, the following are pairwise equivalent:*

(1) *$x$ is contained in $\mathsf{NCR}_{\mathrm{comp}}$.*
(2) *The ratio of $\nu_{\mathrm{opt}}$ to $\mu$ is unbounded (equivalently, $\liminf_{n\to\infty} \nu_{\mathrm{opt}}(x \upharpoonright n)/\mu(x \upharpoonright n)$ diverges to infinity) at $x$ for any computable atomless measure $\mu : 2^{<\mathbb{N}} \to (0,1]$.*
(3) *For any computable acceptable fair odds $O$, there is a left-c.e. $O$-super-martingale $M : 2^{<\mathbb{N}} \to [0,\infty)$ which satisfies $\sup_{\sigma \subsetneq x} M(\sigma) = \infty$.*

Contrary to the case of universally measure zero, the divergence of the infimum limit of the universal prediction $\nu_{\rm opt}(x \upharpoonright n)/\mu(x \upharpoonright n)$ does not characterize the notion of strong measure zero. To see this, it suffices to show the following proposition concerning martingales. A real is called *weakly* 2-*generic* if it is contained in all dense open sets which are c.e. relative to the halting problem $\emptyset'$.

**Proposition 36.** *Let* $\mu : 2^{<\mathbb{N}} \to (0,1]$ *be the outer premeasure defined as* $\mu(\sigma) = 2^{-\lfloor|\sigma|/3\rfloor}$. *Then, for every weakly* 2*-generic real* $g \in 2^{\mathbb{N}}$*, the singleton* $\{g\}$ *is of strong Martin-Löf measure zero, but* $\liminf_{n\to\infty} M(g \upharpoonright n) \leq 1$ *for any* $O_\mu$*-supermartingale* $M$.

*Proof.* We write $O$ for $O_\mu$. Note that $O(\sigma) = 1$ if $|\sigma| \mod 3$ is 0 or 1, and $O(\sigma) = 2$ otherwise. Let $\{(O_s, M_s)\}_{s\in\mathbb{N}}$ be an computable enumeration of all pairs of partial computable acceptable odds and its optimal left-c.e. supermartingale. By $S$ we denote the set of all indices $s$ such that $O_s$ is total. Let $\{N_s\}_{s\in\mathbb{N}}$ be a computable enumeration of all left-c.e. $O$-supermartingales. Note that for any computable $f$, each martingale $M_s$ succeeds on $\{f\}$ for any $s \in S$. Thus, for each $s \in S$ and $n \in \mathbb{N}$ the set

$$\mathcal{D}_{s,n} = \{f \in 2^{\mathbb{N}} : M_s(\sigma) \geq n \text{ for some } \sigma \subset f\}$$

is dense. To see this, for given $\sigma \in 2^{<\mathbb{N}}$, concatenate sufficiently many zeros. Clearly, $\mathcal{D}_{s,n}$ is c.e. open for each $s \in S$ and $n \in \mathbb{N}$. Moreover,

$$\mathcal{E}_{s,n} = \{f \in 2^{\mathbb{N}} : N_s(\sigma) < 1 \text{ for some } \sigma \subset f \text{ of length} \geq n\}$$

is also dense, for each $s, n \in \mathbb{N}$. Since $N_s$ is $O$-supermartingale, for any $\tau \in 2^{<\mathbb{N}}$ with $|\tau| \mod 3 = 0$, there exists $\tau' \in \{0,1\}^3$ such that $N_s(\tau\tau') \leq N_s(\tau)/2$. By iterating this procedure, we easily find a string $\sigma \supset \tau$ with $[\![\sigma]\!] \subseteq \mathcal{E}_{s,n}$. Note that $\mathcal{E}_{s,n}$ is $\emptyset'$-c.e. open for each $s, n \in \mathbb{N}$.

□

## 3. Effectively Closed Sets

In the rest of the paper, we pay attention to effective strong measure zero $\Pi^0_1$ subsets of $2^{\mathbb{N}}$. It is shown that effective strong measure zero and strong Martin-Löf measure zero coincide in the case of $\Pi^0_1$ sets. Moreover, within the class of the $\Pi^0_1$ sets, we characterize the property of effective strong measure zero as a kind of effective perfect set property. We also investigate the Muchnik degrees of effective strong measure zero $\Pi^0_1$ sets. See Simpson [34, 35] for basic facts on the Muchnik degrees of $\Pi^0_1$ sets in Cantor space. We show that any non-zero Muchnik degree of $\Pi^0_1$ sets of effective strong measure zero is incomparable with the Muchnik degrees of the set of Martin-Löf random reals and the set of autocomplex reals. Consequently, if $P \subseteq 2^{\mathbb{N}}$ is a nonempty effective strong measure zero $\Pi^0_1$ set consisting only of noncomputable elements, then some Martin-Löf random real computes no element in $P$, and $P$ has an element that computes no autocomplex real.

3.1. **Combinatorial Theorem.** We first show the following combinatorial theorem. While we use the theorem to characterize closed subsets of $2^{\mathbb{N}}$ of effective strong measure zero in the next subsection, the theorem itself is interesting.

**Theorem 37** (Higuchi/Kihara [21, Theorem 1])**.** *Let $T \subset \mathbb{N}^{<\mathbb{N}}$ be a finitely branching tree. Suppose that $[T] \setminus [\![A]\!]$ is nonempty for any $A \subset T \setminus \{\emptyset\}$ with $\#(A \cap \{0,1\}^{n+1}) \leq \#(T \cap \{0,1\}^n)$ for any $n \in \mathbb{N}$. Then there exists a length-preserving embedding of $2^{<\mathbb{N}}$ into $T$.*

*Proof.* Let $\varphi(T')$ denote the condition that $[T'] \setminus [\![A]\!]$ is nonempty for any $A \subset T' \setminus \{\emptyset\}$ with $\#(A \cap \{0,1\}^{n+1}) \leq \#(T \cap \{0,1\}^n)$ for any $n \in \mathbb{N}$. For $\sigma \in T$, we define $T(\sigma) = \{\tau \in \mathbb{N}^{<\mathbb{N}} : \sigma\tau \in T\}$. It suffices to show that for any $\sigma \in T$ with $\varphi(T(\sigma))$ there exist at least two immediate successors $\sigma i$, $\sigma j$ of $\sigma$ in $T$ with $\varphi(T(\sigma i))$ and $\varphi(T(\sigma j))$.

Fix $\sigma \in T$ with $\varphi(T(\sigma))$. Let $\sigma k_0, \sigma k_1, \cdots, \sigma k_n \in T$ be all immediate successors of $\sigma$ in $T$ with $k_0 < k_1 < \cdots < k_n$. Suppose that there is at most one immediate successor of $\sigma$ in $T$ with the property $\varphi$. Let $i \leq n$ satisfy $\varphi(\sigma k_i)$ if there is such a natural number. For any $j \in \{0, 1, \cdots, n\} \setminus \{i\}$, choose $A_j$ witnessing $\neg\varphi(T(\sigma k_j))$. It is easy to see that $\{k_i\} \cup \bigcup_{j \neq i} \{k_j \tau : \tau \in A_j\}$ witnesses that $\neg\varphi(T(\sigma))$. We have a contradiction. Thus there exist at least two immediate successors of $\sigma$ in $T$ with the property $\varphi$. $\square$

3.2. **Perfect Set Property.** Recall that a *perfect* subset of $2^{\mathbb{N}}$ means a nonempty closed set with no isolated point. This notion is effectivized as follows.

**Definition 38** (Binns [7])**.** A perfect subset $P$ of $2^{\mathbb{N}}$ is said to be *computably perfect* if there exists a computable function $F : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall n \in \mathbb{N})(\forall f \in P)(\exists g \in P)[n \leq |f \cap g| \leq F(n)].$$

A subset of $2^{\mathbb{N}}$ is said to be *diminutive* if it contains no computably perfect subset.

In the case of $\Pi^0_1$ subsets of $2^{\mathbb{N}}$, Binns [7] noticed the following fact.

**Proposition 39** (Binns [7, Lemma 2.4])**.** *If a $\Pi^0_1$ subset of $2^{\mathbb{N}}$ contains a computably perfect subset, then it contains a computably perfect $\Pi^0_1$ subset.*

To show one direction of the equivalence between effective strong measure zero and diminutiveness for closed subsets of $2^{\mathbb{N}}$, we will use the next lemma.

**Lemma 40.** *A perfect subset $P$ of $2^{\mathbb{N}}$ is computably perfect if and only if there exists a computable function $F : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$ and any $f \in P$ there exists an element $g \in P$ such that $F(n) \leq |f \cap g| < F(n+1)$ holds.*

*Proof.* If a computable function $F : \mathbb{N} \to \mathbb{N}$ witnesses that $P$ is computably perfect, then a computable function $F' : \mathbb{N} \to \mathbb{N}$ defined by $F'(0) = 0$ and $F'(n+1) = F(F'(n) + 1)$ witnesses that the latter condition holds.

Conversely, if a computable function $F : \mathbb{N} \to \mathbb{N}$ witnesses that the latter condition holds, then $n \mapsto F(n+1)$ witnesses that $P$ is computably perfect since $n \leq F(n) < F(n+1)$ for any $n \in \mathbb{N}$. $\square$

**Corollary 41** (Higuchi/Kihara [21, Proposition 3])**.** *Every computably perfect subset of* $2^{\mathbb{N}}$ *is not of effective strong measure zero.*

*Proof.* Let $P$ be a computable perfect subset of $2^{\mathbb{N}}$. If a computable function $F$ witnesses the latter condition in Lemma 40 holds, then $\{F(n+1)\}_{n\in\mathbb{N}}$ witnesses that $P$ is not of effective strong measure zero. $\square$

Since effective strong measure zero is preserved under taking subsets, we have the following corollary.

**Corollary 42.** *Every effective strong measure zero subset of* $2^{\mathbb{N}}$ *is diminutive.*

The same argument clearly implies Besicovitch's old result [3, Theorem 2] that every strong measure zero set of reals has no perfect subset. Using Theorem 37, we show that the converse also holds for closed subsets of $2^{\mathbb{N}}$. Although there is no uncountable *closed* set with no *perfect* subset, we know a huge number of uncountable *effectively closed* (i.e., $\Pi^0_1$) sets with no *computably perfect* subsets (see also Binns [5, 6]).

**Theorem 43** (Higuchi/Kihara [21, Theorem 2])**.** *A closed subset of* $2^{\mathbb{N}}$ *is diminutive if and only if it is of effective strong measure zero.*

*Proof.* It remains to show the "only if" part. Fix a closed subset $C$ of $2^{\mathbb{N}}$. Suppose that a computable sequence $\{k_i\}_{i\in\mathbb{N}}$ witnesses that $C$ is not of effective strong measure zero. We show that $C$ contains a computably perfect subset. We may safely assume that $k_i < k_{i+1}$ for all $i \in \mathbb{N}$. Define $F : \mathbb{N} \to \mathbb{N}$ recursively by $F(0) = 0$ and $F(n+1) = F(n) + 2^{k_{F(n)}}$ for all $n \in \mathbb{N}$. Note that $k_{F(n)} \geq n$ and $k_{F(n+1)} - k_{F(n)} \geq 2^{k_{F(n)}}$ for all $n \in \mathbb{N}$. Define $\{T_n\}_{n\in\mathbb{N}}$ by $T_0 = \{\emptyset\}$ and $T_{n+1} = \mathrm{Ext}(C) \cap \{0,1\}^{k_{F(n+1)}}$. Let $T = \bigcup_{n\in\mathbb{N}} T_n$. Since $C$ is closed, note that $C = \bigcap_{n\in\mathbb{N}} [\![T_n]\!]$. Since $C$ is of effective strong measure zero, $C \setminus [\![A]\!]$ is nonempty if $A \subset T \setminus \{\emptyset\}$ satisfies that $\#(A \cap \{0,1\}^{k_{F(n+1)}}) \leq \#(\mathrm{Ext}(C) \cap \{0,1\}^{k_{F(n)}}) \leq 2^{k_{F(n)}}$ for all $n \in \mathbb{N}$. Naturally, $(T, \subset)$ can be seen as a graph of a finitely branching tree and can be embedded into $\mathbb{N}^{<\mathbb{N}}$ so that the image form a finitely branching tree on $\mathbb{N}$ with the assumption of Theorem 37. Thus $2^{<\mathbb{N}}$ has a length-preserving embedding into $(T, \subset)$ by Theorem 37. This implies that $C$ has a computably perfect subset witnessed via $n \mapsto k_{F(n+1)}$. $\square$

Binns [7, Theorem 2.13] showed that a $\Pi^0_1$ subset $P$ of $2^{\mathbb{N}}$ is diminutive if and only if $P$ contains no complex element. We can give another proof of this equivalence using Theorem 4 and Theorem 43 as well as the following theorem.

**Theorem 44.** *A effective strong measure zero* $\Pi^0_1$ *subset of* $2^{\mathbb{N}}$ *is of strong Martin-Löf measure zero.*

*Proof.* Fix a $\Pi^0_1$ subset $P$ of $2^{\mathbb{N}}$ of effective strong measure zero. Given a computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to [0,1]$ and a natural number $n \in \mathbb{N}$, there exists a finite subset $A$ of $2^{<\mathbb{N}}$ such that $\sum_{\sigma\in A}\mu(\sigma) < 2^{-n}$ and $P \subset [\![A]\!]$ since $P$ is of effective strong measure zero and $P$ is compact. Such a finite subset can be found uniformly in $n \in \mathbb{N}$. Thus there exists a computable descending sequence $\{U_n\}_{n\in\mathbb{N}}$ of c.e. open sets such that $X \subset \bigcap_{n\in\mathbb{N}} U_n$ and $\mu^*(U_n) \leq 2^{-n}$ for any $n \in \mathbb{N}$. □

Now we have the following characterization of effective strong measure zero for $\Pi^0_1$ subsets of $2^{\mathbb{N}}$ by Theorem 4, Theorem 43, Theorem 44 and a result from Binns [7, Theorem 2.13].

**Corollary 45.** *For a $\Pi^0_1$ subset $P$ of $2^{\mathbb{N}}$, the following are pairwise equivalent:*

(1) *$P$ is of effective strong measure zero.*
(2) *$P$ is of strong Martin-Löf measure zero.*
(3) *$P$ is diminutive.*
(4) *$P$ contains no complex element.*
(5) *There exists a real $x \in 2^{\mathbb{N}}$ such that no element of $P$* wtt*-computes $x$.*
(6) *The ratio of $\nu_{\mathrm{opt}}$ to $\mu$ is unbounded at any $x \in P$ for any computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$.*
(7) *For any computable acceptable odds $O : 2^{<\mathbb{N}} \to [1,\infty)$, there exists a left-c.e. $O$-supermartingale $M : 2^{<\mathbb{N}} \to [0,\infty)$ such that $M$ succeeds on $P$.*

3.3. **Lattice Operators.** We here see relation between effective strong measure zero and lattice operators.

**Theorem 46.** *For subsets $P$ and $Q$ of $2^{\mathbb{N}}$, $P + Q$ is of effective strong measure zero if and only if so are $P$ and $Q$. The same holds for strong Martin-Löf measure zero.*

*Proof.* Since it is clear that $P + Q$ contains a complex element if and only if so is one of $P$ and $Q$, the proposition holds for strong Martin-Löf measure zero by Theorem 33. By the definition of effective strong measure zero, $P$ and $Q$ are of effective strong measure zero if and only if so are $0P$ and $1Q$. Since $P + Q$ is the disjoint union of $0P$ and $1Q$, $P + Q$ is of effective strong measure zero if and only if so are $0P$ and $1Q$ by Theorem 4. □

**Theorem 47.** *For nonempty subsets $P$ and $Q$ of $2^{\mathbb{N}}$, if $P \times Q$ is of effective strong measure zero, then so are $P$ and $Q$. The same holds for strong Martin-Löf measure zero.*

*Proof.* We just show the case of strong Martin-Löf measure zero. Suppose that one of $P$ and $Q$ is not of strong Martin-Löf measure zero. We may safely assume that $P$ is not of strong Martin-Löf measure zero. Choose a computable atomless outer premeasure $\mu : 2^{<\mathbb{N}} \to [0,1]$ as a witness of this fact. Define a computable atomless outer premeasure $\mu' : 2^{<\mathbb{N}} \to [0,1]$ by $\mu'(\sigma) = \mu(\sigma_0)$ for any $\sigma \in 2^{<\mathbb{N}}$, where $\sigma = \sigma_0 \oplus \sigma_1$. If a subset $A$ of $2^{<\mathbb{N}}$ satisfies $P \times Q \subset [\![A]\!]$, then $B = \{\sigma_0 : (\exists \sigma_1)[\sigma_0 \oplus \sigma_1 \in A]\}$ satisfies $P \subset [\![B]\!]$ and $\sum_{\sigma \in B} \mu(\sigma) \le \sum_{\sigma \in A} \mu'(\sigma)$. Hence $P \times Q$ is not of strong Martin-Löf measure zero. □

We do not know whether the converse of the preceding theorem holds or not. However, if we restrict the statement to $\Pi^0_1$ subsets, then we can prove as we shall see below. To show this, we need the following lemma.

**Lemma 48** (Higuchi/Kihara [21, Lemma 1])**.** *Let $P$ be a $\Pi^0_1$ subset of $2^{\mathbb{N}}$ of effective strong measure zero. Given a computable sequence $\{a_i\}_{i\in\mathbb{N}}$ of naturals, we can (uniformly) find a computable increasing function $F : \mathbb{N} \to \mathbb{N}$ and a computable sequence of finite strings $\sigma_i$, $i \in \mathbb{N}$, of the length $a_i$ such that $[\![\sigma_{F(n)}]\!], [\![\sigma_{F(n)+1}]\!], \cdots, [\![\sigma_{F(n+1)-1}]\!]$ are an open cover of $P$.*

*Proof.* For any $n \in \mathbb{N}$ there are finitely many finite strings $\sigma_{n+i}$, $i \le m$, of the length $a_{n+i}$ such that $\{[\![\sigma_{n+i}]\!]\}_{i\le m}$ is an open cover of $P$, since $P$ is compact and of effective strong measure zero. Moreover, we can find such sequences uniformly in a given $n \in \mathbb{N}$ since $P$ is $\Pi^0_1$. From this, it is clear that the lemma holds. □

**Theorem 49** (Higuchi/Kihara [21, Theorem 3])**.** *If $P$ and $Q$ are $\Pi^0_1$ subsets of $2^{\mathbb{N}}$ of effective strong measure zero, then so is $P \times Q$.*

*Proof.* To show that $P \times Q$ is of effective strong measure zero, fix a computable sequence $\{b_i\}_{i\in\mathbb{N}}$ of naturals. Let $\{a_i\}_{i\in\mathbb{N}}$ be a strictly increasing computable sequence of natural numbers such that $b_i \le 2a_i$ for all $i \in \mathbb{N}$ and, applying Lemma 48 to $P$ and $\{a_i\}_{i\in\mathbb{N}}$, take a computable function $F$ and a computable sequence $\{\sigma_i\}_{i\in\mathbb{N}}$ as in Lemma 48. Here we can safely assume that $F(0) = 0$. Since $Q$ is also of effective strong measure zero, there exist finite strings $\tau_n$, $n \in \mathbb{N}$, of the length $a_{F(n+1)}$ which generate an open cover of $Q$. For each $i \in \mathbb{N}$, define $\rho_i = \sigma_i \oplus (\tau_{n_i} \upharpoonright |\sigma_i|)$, where $n_i$ is the unique natural number such that $F(n_i) \le i < F(n_i + 1)$. Since $|\sigma| = a_i$, we have $|\rho_i| = 2a_i$ for all $i \in \mathbb{N}$.

It suffices to show that $\{\rho_i\}_{i\in\mathbb{N}}$ generates an open cover of $P \times Q$. Fix $f \oplus g \in P \times Q$. Since $g \in Q$, there exists $n \in \mathbb{N}$ such that $\tau_n \subset g$. By the choice of finite strings $\sigma_i$, $F(n) \le i < F(n+1)$, there exists $m$ with $F(n) \le m < F(n+1)$ such that $\sigma_m \subset f$. We have $\rho_m \subset f \oplus g$ and, thus, $P \times Q \subset \bigcup_{i\in\mathbb{N}} [\![\rho_i]\!]$. □

**Corollary 50** (Higuchi/Kihara [21, Corollary 2])**.** *For nonempty $\Pi^0_1$ subsets $P$ and $Q$ of $2^{\mathbb{N}}$, the following are pairwise equivalent:*

(1) *$P$ and $Q$ are of effective strong measure zero.*
(2) *$P + Q$ is of effective strong measure zero.*
(3) *$P \times Q$ is of effective strong measure zero.*

3.4. **Closure Properties.** Some closure properties of a concept for $\Pi^0_1$ subsets of $2^{\mathbb{N}}$ are sometimes useful when we study degree structures of nonempty $\Pi^0_1$ subsets of $2^{\mathbb{N}}$. It is straightforward to see that effective strong measure zero is closed under taking subsets. We shall see that effective strong measure zero is also closed under taking the images of computable functions for $\Pi^0_1$ subsets of $2^{\mathbb{N}}$.

For a partial computable function $\Phi$ on $2^{\mathbb{N}}$, a finite binary string $\sigma$ and a natural number $n$, we use $\Phi(\sigma; n)$ to denote the computation of $\Phi$ with an oracle $\sigma$, an input $n$ and step $|\sigma|$ and we use $\Phi(\sigma)$ to denote the finite string $\tau$ of the maximum length such that $\Phi(\sigma; n) = \tau(n)$ for all $n < |\tau|$.

**Theorem 51** (Higuchi/Kihara [21, Theorem 4]). *The image $\Phi(P)$ of a $\Pi^0_1$ subset $P$ of $2^{\mathbb{N}}$ of effective strong measure zero under a computable function $\Phi : P \to 2^{\mathbb{N}}$ is again of effective strong measure zero.*

*Proof.* Fix a $\Pi^0_1$ subset $P$ of $2^{\mathbb{N}}$ of effective strong measure zero and a computable function $\Phi : P \to 2^{\mathbb{N}}$, and assume, contrary to our theorem, $\Phi(P)$ is not of effective strong measure zero. Let a computable sequence $\{k_i\}_{i\in\mathbb{N}}$ of naturals be a witness of this assumption. Since $P$ is $\Pi^0_1$, we can find a computable sequence $\{k'_i\}_{i\in\mathbb{N}}$ of naturals such that $|\sigma| \geq k'_i$ implies $|\Phi(\sigma)| \geq k_i$ for all $\sigma \in \mathrm{Ext}(P)$. Using the effective strong measure zero of $P$, choose a sequence of finite strings $\sigma_i$ of length $k'_i$ such that $P \subset \bigcup_{i\in\mathbb{N}} [\![\sigma_i]\!]$. We have an open cover $\{[\![\Phi(\sigma_i)]\!]\}_{i\in\mathbb{N}}$ of $\Phi(P)$, contradicting our assumption that $\{k_i\}_{i\in\mathbb{N}}$ witnesses that $\Phi(P)$ is not of effective strong measure zero. Thus $\Phi(P)$ is of effective strong measure zero. $\square$

A subset $X$ of $\mathbb{N}^{\mathbb{N}}$ is called *computably bounded (c.b.)* if there is a computable function $F : \mathbb{N} \to \mathbb{N}$ with $g(n) < F(n)$ for any $g \in X$ and $n \in \mathbb{N}$. It is well-known that every c.b. $\Pi^0_1$ set is computably homeomorphic to a $\Pi^0_1$ subset of $2^{\mathbb{N}}$.

**Remark 52.** We can extend Definition 1 and Definition 38 to c.b. subsets of $\mathbb{N}^{\mathbb{N}}$ in the straightforward way. Also, Theorem 43, Corollary 50 and Theorem 51 can be easily extended to c.b. closed or $\Pi^0_1$ subsets of $\mathbb{N}^{\mathbb{N}}$. We shall use these extended theorems later.

**Theorem 53** (Simpson [34, Theorem 4.7]). *Let $P \subset \mathbb{N}^{\mathbb{N}}$ be a nonempty c.b. $\Pi^0_1$ set and let $\Phi : P \to \mathbb{N}^{\mathbb{N}}$ be a computable function. Then $\Phi(P)$ is a nonempty c.b. $\Pi^0_1$ subset of $\mathbb{N}^{\mathbb{N}}$.*

**Theorem 54** (Simpson [34, Lemma 6.9]). *Let $M \subset \mathbb{N}^{\mathbb{N}}$ and let $P \subset \mathbb{N}^{\mathbb{N}}$ be a nonempty c.b. $\Pi^0_1$ set. If $M \leq_{\mathrm{w}} P$, then $P$ contains a nonempty c.b. $\Pi^0_1$ subset $Q$ with $M \leq_{\mathrm{s}} Q$.*

We prove the following theorem using the technique of the proof of Corollary 2.16 in Binns [5].

**Theorem 55** (Higuchi/Kihara [21, Theorem 7]). *Let $\mathfrak{A}$ be a set of nonempty c.b. $\Pi^0_1$ subsets of $\mathbb{N}^{\mathbb{N}}$ which is closed under taking nonempty $\Pi^0_1$ subset and* *taking the images of computable functions. Let $P \subset \mathbb{N}^{\mathbb{N}}$ and let $Q \in \mathfrak{A}$. If $P \leq_{\mathrm{w}} Q$, then some subset of $P$ is in $\mathfrak{A}$.*

*Proof.* Suppose that $P \leq_{\mathrm{w}} Q$. By Theorem 54, there exists a computable function $\Phi : Q' \to P$ for some nonempty $\Pi^0_1$ subset $Q'$ of $Q$. The image $\Phi(Q')$ is a nonempty c.b. $\Pi^0_1$ subset of $P$ by Theorem 53. We have $\Phi(Q') \in \mathfrak{A}$ by the closure properties of $\mathfrak{A}$. □

Applying the theorem to $\mathfrak{A}$ as the set of all nonempty $\Pi^0_1$ subsets of $2^{\mathbb{N}}$ of effective strong measure zero, we have the following corollary.

**Corollary 56** (Higuchi/Kihara [21, Corollary 3]). *If a subset $P$ of $2^{\mathbb{N}}$ is Muchnik reducible to a nonempty $\Pi^0_1$ subset of $2^{\mathbb{N}}$ of effective strong measure zero, then $P$ contains a nonempty $\Pi^0_1$ subset of effective strong measure zero.*

3.5. **MLR and DNC.** We denote MLR the set of all Martin-Löf random elements of $2^{\mathbb{N}}$ and denote DNC the set of all diagonally non-computable elements of $\mathbb{N}^{\mathbb{N}}$. Here a real $f \in \mathbb{N}$ is *Martin-Löf random* if $\{f\}$ is not of Martin-Löf $\lambda$-zero, where $\lambda$ denotes the fair-coin measure, and a real $f \in \mathbb{N}$ is *diagonally non-computable* if $f(e) \neq \Phi_e(e)$ holds for any $e \in \mathbb{N}$, where $\{\Phi_e\}_{e\in\mathbb{N}}$ is a fixed standard effective enumeration of all computable partial function from $\mathbb{N}$ to $\mathbb{N}$. Note that the Muchnik degree of the diagonally non-computable functions can be characterized in terms of Kolmogorov complexity, since a real $x \in 2^{\mathbb{N}}$ is autocomplex if and only if it computes a diagonally non-computable function.

Simpson [34] proved that MLR is Muchnik incomparable with any perfect thin $\Pi^0_1$ subset of $2^{\mathbb{N}}$. We use the technique of his proof to show that MLR and DNC are incomparable with any nonempty $\Pi^0_1$ subset of $2^{\mathbb{N}}$ of effective strong measure zero with no computable element. We use the facts that every nonempty $\Pi^0_1$ subset of MLR is Muchnik equivalent to MLR and that MLR contains a nonempty $\Pi^0_1$ subset. See [34].

**Theorem 57.** *Let $P \subset$ MLR be a nonempty $\Pi^0_1$ set and let $\Phi : P \to \mathbb{N}^{\mathbb{N}}$ be a computable function. If $\Phi(P)$ contains no computable element, then $\Phi(P) \equiv_{\mathrm{w}} P$.*

*Proof.* By Simpson [34, Corollary 4.9], we know that $\Phi(f) \leq_{\mathrm{tt}} f$ for all $f \in P$, where $\leq_{\mathrm{tt}}$ refers to the *truth-table reducibility*. Additionally, by Demuth [11, Lemma 30], we know that, for any $f \in P$, $\Phi(f)$ is Turing equivalent to an element of MLR. Thus MLR $\leq_{\mathrm{w}} \Phi(P) \leq_{\mathrm{w}} P \equiv_{\mathrm{w}}$ MLR holds. Hence $\Phi(P) \equiv_{\mathrm{w}} P$. □

**Theorem 58** (Higuchi/Kihara [21, Theorem 9]). *Let $\mathfrak{A}$ be a set of nonempty c.b. $\Pi^0_1$ subsets of $\mathbb{N}^{\mathbb{N}}$ which is closed under taking nonempty $\Pi^0_1$ subset and taking the images of computable functions. Let $P \subset \mathbb{N}^{\mathbb{N}}$ be Muchnik reducible to* MLR *and let $Q \in \mathfrak{A}$ contain no computable element. Suppose that every c.b. $\Pi^0_1$ subset of $P$ is not in $\mathfrak{A}$. Then $P$ and $Q$ are Muchnik incomparable.*

*Proof.* Since $P \leq_{\mathrm{w}} Q$ implies that $P$ contains a nonempty $\Pi^0_1$ subset in $\mathfrak{A}$ by Theorem 55, we have $P \not\leq_{\mathrm{w}} Q$. Suppose that $Q \leq_{\mathrm{w}} P$. Since $P \leq_{\mathrm{w}}$ MLR, we have $Q \leq_{\mathrm{w}}$ MLR. Choose a nonempty $\Pi^0_1$ set $R \subset$ MLR and a computable function $\Phi : R \to Q$. By Theorem 54 and Theorem 57, we have $R \equiv_{\mathrm{w}} \Phi(R)$. By the closure properties of $\mathfrak{A}$, we have $\Phi(R) \in \mathfrak{A}$. On the other hand, we have $P \leq_{\mathrm{w}} R \equiv_{\mathrm{w}} \Phi(R)$. A contradiction. Thus $Q \not\leq_{\mathrm{w}} P$. □

**Proposition 59.** *Every nonempty* $\Pi^0_1$ *subset of* MLR *contains a computably perfect subset.*

*Proof.* By Simpson [34, Lemma 8.9], every nonempty $\Pi^0_1$ subset of MLR is of positive measure. By Hertling [19, Proposition 8], we know that any closed subset of $2^{\mathbb{N}}$ of positive measure contains a computably perfect subset. Thus the proposition holds. □

Applying Theorem to $P =$ MLR and $\mathfrak{A}$ as the set of all nonempty c.b. $\Pi^0_1$ subsets of $\mathbb{N}^{\mathbb{N}}$ of effective strong measure zero, we have the following corollary.

**Corollary 60** (Higuchi/Kihara [21, Corollary 4])**.** *Let $Q$ be an* $\Pi^0_1$ *subset of* $2^{\mathbb{N}}$ *of effective strong measure zero with no computable element. Then $Q$ is Muchnik incomparable with* MLR.

**Proposition 61.** *Every nonempty c.b.* $\Pi^0_1$ *subset of* DNC *is computably perfect.*

*Proof.* Let $P \subset$ DNC be a nonempty c.b. $\Pi^0_1$ set. Using Recursion Theorem, given a $\sigma \in 2^{<\mathbb{N}}$, we can effectively find $n_\sigma$ such that $\{n_\sigma\}(n_\sigma)$ is the unique value of $f(n_\sigma)$ for some $s \in \mathbb{N}$ and some $f \in P_s \cap [\![\sigma]\!]$ (if exist) such that, for any $f, g \in P_s \cap [\![\sigma]\!]$, $f(n_\sigma) = g(n_\sigma)$, where $P_s$ is a clopen set which is the $s$-th approximation of $P$. Then the computable function $m \mapsto \max\{n_\sigma : \sigma \in 2^{<\mathbb{N}} \ \& \ |\sigma| = m\}$ witnesses that $P$ is computably perfect. □

It is well-known that DNC $\leq_{\mathrm{w}}$ MLR. See, for instance, Giusto/Simpson [18, Lemma 6.18]. Thus applying the theorem to $P =$ DNC and $\mathfrak{A}$ as the set of all nonempty c.b. $\Pi^0_1$ subsets of $\mathbb{N}^{\mathbb{N}}$ of effective strong measure zero, we have the following corollary.

**Corollary 62** (Higuchi/Kihara [21, Corollary 5])**.** *Let $Q$ be a* $\Pi^0_1$ *subset of* $2^{\mathbb{N}}$ *of effective strong measure zero with no computable element. Then $Q$ is Muchnik incomparable with* DNC.

**Remark 63.** Indeed, one direction of Corollary 62, i.e., DNC is Muchnik reducible to no diminutive $\Pi^0_1$ subset of $2^{\mathbb{N}}$, can be obtained easily using Theorem 2.12 and Corollary 2.14 of Binns [7].

**Remark 64.** By Binns [7, Theorem 3.8, Theorem 3.9] and Binns [6], we have that thinness or smallness imply diminutiveness. Thus Corollary 60 and Corollary 62 hold even when we replace the property "of effective strong measure zero" with the properties "thin" or "small". Here, Simpson [34, Theorem 9.15] showed that MLR is Muchnik incomparable with any nonempty thin perfect $\Pi^0_1$ subset of $2^{\mathbb{N}}$.

## 4. Smallness and Kolmogorov Complexity

As is well known, if a nonempty $\Pi^0_1$ subset of Cantor space consists only of noncomputable elements, then it must be perfect. As seen in Corollary 45 (Binns [7]), a nonempty $\Pi^0_1$ set has a complex element if and only if it has a computably perfect subset (or equivalently, it is not of effective strong measure zero). Then, if a $\Pi^0_1$ set is not of effective strong measure zero, does it contain a real all of whose initial segments are sufficiently complex? Conversely, does every nonempty effective strong measure zero $\Pi^0_1$ set have an anticomplex element? In this section, we construct two counterexamples related to the above two questions.

4.1. **Nonbasis Theorem.** A $\Pi^0_1$ subset $P$ of $2^{\mathbb{N}}$ is *small* [5] if $\mathrm{Brl}(P)$ is not dominated by any computable function. A $\Pi^0_1$ subset $P$ of $2^{\mathbb{N}}$ is *very small* [5] if $\mathrm{Brl}(P)$ dominates all computable functions. An infinite binary string $f \in 2^{\mathbb{N}}$ is 1-*generic* if for any c.e. open set $U$, either $f \in U$ or $[\![\sigma]\!] \cap U = \emptyset$ for some $\sigma \subsetneq f$. The previous works on measure theoretic smallness of $\Pi^0_1$ sets implies the following non-basis theorem:

**Theorem 65** (Small Non-Basis Theorem [21, Theorem 10])**.** *If a small $\Pi^0_1$ set $P \subseteq 2^{\mathbb{N}}$ contains no computable element, then we have the following:*

(1) *No element of $P$ is complex.*
(2) *No element of $P$ is computable in a 1-generic real.*

*Proof.* By Binns [6, 7], every small $\Pi^0_1$ set is diminutive. Moreover, by Cenzer/Kihara/Weber/Wu [10, Theorem 3.5], every such small $\Pi^0_1$ set is immune. By Binns [7, Theorem 2.13], every element of diminutive $\Pi^0_1$ set is non-complex. By Demuth/Kučera [12], every 1-generic real computes no element of an immune $\Pi^0_1$ set. □

**Definition 66.** A real $x \in 2^{\mathbb{N}}$ is *infinitely often complex* [22], abbreviated as *i.o. complex*, if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{K}(x \upharpoonright f(n)) \geq n$ for infinitely many $n \in \mathbb{N}$. A real $x \in 2^{\mathbb{N}}$ is *anti-complex* [15] if it is not i.o. complex. A real $x \in 2^{\mathbb{N}}$ is K-*trivial* if there is a constant $c \in \mathbb{N}$ such that $\mathrm{K}(x \upharpoonright n) \leq \mathrm{K}(n) + c$ for any $n \in \mathbb{N}$. A real $x \in 2^{\mathbb{N}}$ is *infinitely often* K-*trivial* [2], abbreviated as *i.o.* K-*trivial*, if there is a constant $c \in \mathbb{N}$ such that $\mathrm{K}(x \upharpoonright n) \leq \mathrm{K}(n) + c$ for infinitely many $n \in \mathbb{N}$. A real $x \in 2^{\mathbb{N}}$ is *Schnorr trivial* [13] if, for any computable measure machine $M$, there is a computable measure machine $N$ and a constant $c \in \mathbb{N}$ such that $\mathrm{K}_N(x \upharpoonright n) \leq \mathrm{K}_M(n) + c$ for any $n \in \mathbb{N}$.

Kjos-Hanssen/Merkle/Stephen [26] showed that a real $x \in 2^{\mathbb{N}}$ is complex if and only if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$\mathrm{K}(x \upharpoonright f(n)) \geq n$ holds for any $n \in \mathbb{N}$. Consequently, every complex real is i.o. complex. Franklin/Stephan [17] showed that a real $x$ is Schnorr trivial if and only if it is *computably* tt-*traceable*, i.e., there is an order $h$ such that, for every $g \leq_{\mathrm{tt}} x$, there is a computable trace $\{T_n\}_{n\in\mathbb{N}}$ with bound $h$ and $g \in T_n$ for each $n \in \mathbb{N}$. Moreover, Franklin/Greenberg/Stephan/Wu [15] showed that a real $x$ is anti-complex if and only if it is *c.e.* wtt-*traceable*, i.e., there is an order $h$ such that, for every $g \leq_{\mathrm{wtt}} x$, there is a c.e. trace $\{T_n\}_{n\in\mathbb{N}}$ with bound $h$ and $g \in T_n$ for each $n \in \mathbb{N}$.

**Theorem 67** (Very Small Non-Basis Theorem, see also Binns/Kjos-Hanssen [8])**.** *If a very small $\Pi^0_1$ set $P \subseteq 2^{\mathbb{N}}$ contains no computable element, then we have the following:*

1. *Every element of $P$ is Schnorr trivial. In particular, every element of $P$ is anti-complex.*
2. *No element of $P$ is computable in a 1-generic real.*

*Proof.* It suffices to show that every element $x \in P$ is computably tt-traceable. Fix $g \leq_{\mathrm{tt}} x$. Then there is a total computable functional $\Psi$ and a computable order $h$ such that $g(n) = \Psi^{x\upharpoonright h(n)}(n)$ for any $n \in \mathbb{N}$. As $P$ is very small, the function $n \mapsto \#\{x \upharpoonright h(n) : x \in P\}$ is majorized by a computable function $h^\star$. Let $\{P_s\}_{s\in\mathbb{N}}$ be a computable approximation of $P$, and then, for each $n \in \mathbb{N}$, wait for stage $s(n)$ when $\#\{x \upharpoonright h(n) : x \in P_{s(n)}\} \leq h^\star(n)+1$. Then, $\#\{\Psi^{x\upharpoonright h(n)}(n) : x \in P_{s(n)}\}$ is also bounded by $h^\star(n) + 1$. Moreover, $\{\Psi^{x\upharpoonright h(n)}(n) : x \in P_{s(n)}\}_{n\in\mathbb{N}}$ is a computable trace, since $\Psi$ is a total computable functional.

□

Note that Franklin [16] showed that there is a 1-generic real which is Turing equivalent to a Schnorr trivial real.

4.2. **A Perfect Set which is Not Small.** The Small Non-Basis Theorem 65 may have some applications. Barmpalias/Vlek [2] showed that, if a real is computable in a 1-generic, then it is i.o. K-trivial; and there is a $\Pi^0_1$ subset of $2^{\mathbb{N}}$ consisting of i.o. K-trivial reals but does not contain any K-trivial reals. First we construct a perfect set consisting of non-generic reals which are both complex and i.o. K-trivial. Here, note that, if a reals is complex, then it is not K-trivial.

**Theorem 68.** *There is a perfect $\Pi^0_1$ set $P \subseteq 2^{\mathbb{N}}$ which satisfies the following:*

1. *Every element of $P$ is i.o.* K*-trivial.*
2. *Every element of $P$ is complex.*
3. *No element of $P$ is computable in a 1-generic real.*

*Proof.* It suffices to construct a computably perfect immune $\Pi^0_1$ set of reals which are i.o. K-trivial and tt-DNC. Here, a real $x$ is tt-DNC if there is a function $f \leq_{\mathrm{tt}} x$ which is *diagonally noncomputable*, i.e., for every $e \in \mathbb{N}$, if $\varphi_e(e)$ converges, then $f(e) \neq \varphi_e(e)$ holds. Kjos-Hanssen/Markle/Stephan [26] showed that a real is complex if and only if it is tt-DNC.

**Claim 69.** *For some computable order $h$, we have an infinite computable tree $T_0 \subseteq 2^{<\mathbb{N}}$ such that every infinite path through $T_0$ is i.o.* K*-trivial, and every string $\sigma \in T_0$ of length $h(n)$ has at least three extensions $\tau \in T_0$ of length $h(n+1)$, for each $n \in \mathbb{N}$.*

*Proof.* As pointed out by Barmpalias-Vlek [2, Lemma 2.6], there is a c.e. dense set $V$ of strings $\sigma$ such that $\mathrm{K}(\sigma) \leq \mathrm{K}(|\sigma|) + d$ (by concatenating many zeros). For all $\sigma \in \{0,1\}^n$, wait for a string $\tau_\sigma$ extending $\sigma$ to be enumerated into $V$. Then, $g(n)$ is defined to be $\max\{|\tau_\sigma| : \sigma \in \{0,1\}^n\}$. Now, we assume that $h(n)$ has been already defined, and $T_0 \cap \{0,1\}^{\leq h(n)}$ has been already determined. For each string $\sigma \in T_0$, there are at least four extensions $\tau_{\sigma 00}, \tau_{\sigma 01}, \tau_{\sigma 10}, \tau_{\sigma 11}$ of length $\leq g(h(n)+2)$. Define $h(n+1) = g(h(n)+2)$, extend all $\tau_{\sigma ij}$ for $i, j < 2$ to strings of length $h(n+1)$, and enumerate them into $T_0$.

□

Without loss of generality, we may assume that $T_0$ has at least three string of length $h(0)$. For each $e \in \mathbb{N}$, if there is a string $\tau \in W_e$ of length $\geq h(e)$, then define $\sigma_e = \tau \upharpoonright h(e)$. Otherwise, $\sigma_e$ is undefined. Let $V$ be the set of all indices $e$ such that $\sigma_e$ is defined. Then, define $T_1$ to be $T_1 = T_0 \setminus \bigcup_{e \in V} [\![\sigma_e]\!]$. For each $e \in \mathbb{N}$, let $D$ be the set of all indices $e$ such that $\varphi_e(e)$ converges and $|\varphi_e(e)| = h(e)$. Then, define $T_2$ to be $T_2 = T_1 \setminus \bigcup_{e \in D} [\![\varphi_e(e)]\!]$. Clearly, $T_2$ is co-c.e.

**Claim 70.** *$T_2$ is infinite.*

*Proof.* At most two strings of each length $h(n)$ can be removed from $T_0$, while each string in $T_0$ of length $h(n)$ must have at least three extensions in $T_0$ of length $h(n+1)$.

□

**Claim 71.** *$[T_1]$ is immune, hence $[T_2]$ is also immune.*

*Proof.* If $W_e$ is infinite, then there is a string $\tau \in W_e$ of length greater than $h(e)$. Hence, $\sigma_e$ must be defined. By the definition of $T_1$, the string $\sigma_e$ has no extension in $[T_1]$.

□

**Claim 72.** *For every infinite path $x$ through $T_2$, the function $n \mapsto x \upharpoonright h(n)$ is diagonally noncomputable. In particular, every infinite path through $T_2$ is* tt-*DNC.*

*Proof.* If $\varphi_e(e)$ is defined to be $x \upharpoonright h(e)$, then, by our definition of $T_2$, the string $x \upharpoonright h(e)$ must be removed from $T_2$. Note that the function $n \mapsto x \upharpoonright h(n)$ is tt-reducible to $x$, since $h$ is computable. Thus, such $x$ must be tt-DNC.

□

Consequently, $P = [T_2]$ is an immune $\Pi^0_1$ set consisting of reals which are i.o. K-trivial and tt-DNC. By Demuth/Kučera [12] and the immunity of $P$, the set $P$ has no element computable in a 1-generic.

□

4.3. **Small, but Not Very Small.** As mentioned by Barmpalias/Vlek [2], if a real is not complex, then it is i.o. K-trivial. Next we construct a perfect set consisting of non-generic reals which are i.o. complex, but not complex.

**Theorem 73.** *There is a perfect $\Pi^0_1$ set $P \subseteq 2^{\mathbb{N}}$ which satisfies the following:*

1. *No element of $P$ is complex.*
2. *Every element of $P$ is i.o. complex.*
3. *No element of $P$ is computable in a 1-generic real.*

*Proof.* It suffices to construct a small $\Pi^0_1$ set consisting of reals which are not c.e. wtt-traceable. Here, recall that a real $x$ is c.e. wtt-traceable if there is a computable order $b$ such that, for every $f \leq_{\rm wtt} x$, there is a c.e. trace $\{T_n\}_{n\in\mathbb{N}}$ with bound $b$ such that $f(n) \in T_n$ holds for any $n \in \mathbb{N}$. Note that the bound $b$ can be replaced by any computable order. Hereafter, we fix a computable order $b$ such that the sequence $\{b(n+1) - b(n)\}_{n\in\mathbb{N}}$ is non-decreasing and unbounded, and let $B_n$ be the half-open interval $[b(n), b(n+1))$ for each $n \in \mathbb{N}$. Recall that a real is c.e. wtt-traceable if and only if it is anti-complex.

**Claim 74.** *Assume that, for every partial computable function $\psi$, there is $u \in \mathbb{N}$ such that $\psi(n) \neq x \upharpoonright n$ holds for all $n \in B_u$. Then, $x$ is i.o. complex.*

*Proof.* Otherwise, $x$ is c.e. wtt-traceable. Then, as the function $u \mapsto x \upharpoonright b(u+1)$ is wtt-reducible to $x$, there is a c.e. trace $\{T_u\}_{u\in\mathbb{N}}$ with bound $\#B_u$. Hence, we have a partial computable function $\varphi$ defining $\{T_u\}_{u\in\mathbb{N}}$ in the sense that $T_u$ is the set consisting of $\varphi(n)$ with $n \in B_u$. Then, for each $u \in \mathbb{N}$, there is $n \in B_u$ with $\varphi(n) = x \upharpoonright b(u+1)$. However, this implies $\psi(n) = \varphi(n) \upharpoonright n = x \upharpoonright n$.

□

*Requirements.* We need to construct a $\Pi^0_1$ set $P = [T_P] \subseteq 2^{\mathbb{N}}$ satisfying the following trace-avoiding requirements $\{\mathcal{T}_e\}_{e\in\mathbb{N}}$ and smallness requirements $\{\mathcal{S}_e\}_{e\in\mathbb{N}}$:

$$\mathcal{T}_e : (\exists u)\ [(\forall n \in B_u)\ |\Phi_e(n)| = n \Rightarrow (\forall n \in B_u)\ \Phi_e(n) \notin T_P],$$

$$\mathcal{S}_e : \Phi_e \text{ total unbounded} \Longrightarrow (\exists n)\ [\Phi_e(n), \Phi_e(n+1)] \cap \mathrm{Brl}(P) = \emptyset.$$

Here, $\{\Phi_e\}_{e\in\mathbb{N}}$ is an effective enumeration of all partial computable functions, and $[l, r]$ denotes the interval $\{m : l \leq m \leq r\}$. The *priority ordering* is defined as $\mathcal{T}_n < \mathcal{S}_n < \mathcal{T}_{n+1}$ for any $n \in \mathbb{N}$. For strategies $\mathcal{Q}$ and $\mathcal{R}$, if $\mathcal{Q} < \mathcal{R}$, then $\mathcal{R}$ is said to be *lower priority strategy* than $\mathcal{Q}$, and $\mathcal{Q}$ is *higher priority strategy* than $\mathcal{R}$.

*Strategy $\mathcal{T}_e$.* This strategy may have a parameter $u(e, s)$, at each stage $s \in \mathbb{N}$. At stage $s + 1$, if $u(e, s)$ is undefined, then choose sufficiently large $u(e, s + 1) \in \mathbb{N}$ such that any element contained in $B_{u(e,s+1)}$ has not been mentioned in our construction. If $u(e, s)$ has been already defined, then set

$u(e, s+1) = u(e, s)$. Then each string $\sigma$ of length $\leq \max B_{u(e,s+1)}$ will be protected from any trimming action by all lower priority strategies. If $\Phi_e(n)$ converges and $|\Phi_e(n)| = n$ holds for some $n \in B_{u(e,s+1)}$, the strategy $\mathcal{T}_e$ removes the string $\Phi_e(n)$ from $P$. Formally, define $P_{s+1}$ as follows:

$$P_{s+1} = P_s \setminus \bigcup \{[\![\Phi_e(n)]\!] : \Phi_e(n) \downarrow \text{ and } |\Phi_e(n)| = n \in B_{u(e,s+1)}\}.$$

*Strategy $\mathcal{S}_e$.* This strategy may have a parameter $l_{e,s}$, at each stage $s \in \mathbb{N}$. At stage $s+1$, if $l_{e,s}$ is undefined, then choose sufficiently large $l_{e,s+1} \in \mathbb{N}$ which has not been mentioned in our construction. If $l_{e,s}$ has been already defined, then set $l_{e,s+1} = l_{e,s}$. Wait for $l_{e,s+1} < \Phi_{e,s+1}(n) \downarrow \leq \Phi_{e,s+1}(n+1) \downarrow$ for some $n \in \mathbb{N}$. Here, $\Phi_{e,s+1}(n) \downarrow = y$ denotes the $e$-th partial computable function halts by stage $s+1$, and outputs $y \in \mathbb{N}$. If it does not happen, then the strategy $\mathcal{S}_e$ makes no action at this stage. Whenever it happens, for each string $\sigma$ of length $\Phi_{e,s+1}(n)$, choose the living leftmost string $L(\sigma)$ of length $\Phi_{e,s+1}(n+1)$ extending $\sigma$. Then, by the trimming action of the strategy $\mathcal{S}_e$, all strings which extend some strings $\sigma$ of length $\Phi_{e,s+1}(n)$ but are incomparable with $L(\sigma)$ are removed from $P$. Formally, define $P_{s+1}$ as follows:

$$P_{s+1} = P_s \setminus \bigcup \{[\![\tau]\!] : (\exists \sigma)\ |\sigma| = \Phi_{e,s+1}(n)\ \&\ \sigma \subset \tau \mid L(\sigma)\}.$$

After the action, the strategy $\mathcal{S}_e$ *injures all lower priority strategies* by forcing all parameters of lower priority strategies to be undefined.

**Claim 75.** *$P = \bigcap_s P_s$ is nonempty.*

*Proof.* Each strategy chooses some intervals of heights: the $\mathcal{T}_e$ strategy chooses an interval $B_{u(e,s)}$; and the $\mathcal{S}_e$ strategy chooses $[\Phi_{e,s}(n), \Phi_{e,s}(n+1)]$. Eventually, these intervals are pairwise disjoint. For each such $n \in B_{u(e,s)}$, at most one string of length $n$ is removed from $P$ by the trimming action executed by each $\mathcal{K}_c$ strategy. By the action of $\mathcal{S}_e$ strategy, some string of length in $[\Phi_{e,s}(n), \Phi_{e,s}(n+1)]$ also survives.

□

**Claim 76.** *Every strategy is injured at most finitely often.*

*Proof.* Inductively assume that some strategy is never injured after some stage $s$. Then, by our construction, the $\mathcal{T}$-strategies injure no other strategies, and each $\mathcal{S}$-strategy can act and injure lower priority strategies at most one time, after stage $s$.

□

**Claim 77.** *The requirements $\mathcal{T}_e$ are satisfied.*

*Proof.* Since $\mathcal{T}_e$ is injured at most finitely often, $u(e) = \lim_s u(e, s)$ converges. By the action of $\mathcal{T}_e$ strategy, if $|\Phi_e(n)| = n \in B_{u(e)}$, then $\Phi_e(n)$ is removed from $P$.

□

**Claim 78.** *The requirements $\mathcal{S}_e$ are satisfied.*

*Proof.* Since $\mathcal{S}_e$ is injured at most finitely often, $l^\star = \lim_s l_{e,s}$ converges. If $\Phi_e$ is total and unbounded, then $l^\star < \Phi_e(n) \downarrow$ must happens for some $n \in \mathbb{N}$. If $\Phi_e(n+1) < \Phi_e(n)$, then there is no problem. Assume that $\Phi_e(n) \leq \Phi_e(n+1)$. By the action of $\mathcal{S}_e$ strategy, every string $\sigma \in P$ of length $\Phi_e(n)$ has just one extension in $P$ of length $\Phi_e(n+1)$. In other words, $[\Phi_e(n), \Phi_e(n+1)] \cap \mathrm{Brl}(P) = \emptyset$.

□

By Claim 74, the $\mathcal{T}$-strategies ensure that every element of $P = \bigcap P_s$ is i.o. complex, and the $\mathcal{S}$-strategies ensure that $P$ is small, by Binns [6, Theorem 2.10]. Note that every i.o. complex reals is not computable. Hence, by Small Basis Theorem 65, every element of $P$ is neither complex nor computable in a 1-generic real, as desired.

□

**Remark 79.** As proved by Binns [5], there is a Muchnik degree that contains a small $\Pi^0_1$ set but no very small $\Pi^0_1$ set in $2^{\mathbb{N}}$. By combining with Very Small Nonbasis Theorem 67, our previous proof provides an alternative proof for Binns' result.

## 5. Conclusion

Our underlying idea is inspired by the basic observation from algorithmic randomness theory that *a real is captured by an effectively-small set if and only if it is "effective-ish."* Concretely speaking, a real $x$ is captured by an effectively null set if and only if it is not algorithmically random (i.e., effectively compressible, or equivalently, effectively predictable), and it is captured by a set of effective Hausdorff (resp., packing) dimension $s$ if and only if its compression ratio $\liminf \mathrm{K}(x \upharpoonright n)/n$ (resp., $\limsup \mathrm{K}(x \upharpoonright n)/n$) is less than or equal to $s$.

Therefore, algorithmic randomness theory is sometimes viewed as measure (dimension) theory of arbitrary "*individual reals*" (but not a theory of "sets of reals"). In this way, an effectivization of measure-theoretic or set-theoretic smallness is clearly related to an effective property of individual reals. Of course, there are many set-theoretic smallness notions other than strong measure zero. This motivates us to study the "set theory of individual reals." The typical question is: *given a smallness property $\mathcal{P}$ (in set theory), is there a degree-theoretic (or randomness-theoretic) characterization of "effectively $\mathcal{P}$"?*

More specifically, one may consider the case where $\mathcal{P}$ is chosen as being Hausdorff (packing) $h$-null for all dimension functions $h$. Of course, our research has followed this thread, because a set is of (effectively) strong measure zero if and only if it is Hausdorff $h$-null for all (computable) dimension functions $h$. Recently, Kihara and Miyabe [25] applied a result on strong measure zero to characterize a randomness-theoretic lowness property for individual reals. Moreover, they pointed out the relationship between Binns' notion of very smallness and the notion of effective packing $h$-nullness for all computable dimension functions $h$.

One can also refine our typical question described above. Model-theoretically, the concept of "effectively $\mathcal{P}$" could be rephrased as the relativized concept $\mathcal{P}^{\mathcal{U}}$ in the *computable universe* $\mathcal{U}$. By generalizing this idea, one can also study the concept of $\mathcal{P}^{\mathcal{M}}$ for an inner model $\mathcal{M}$, in $V$, of a weak system. For instance, it is natural to ask about the relationship between the $\alpha$-degree structure and the property $\mathcal{P}^{L_\alpha}$ for an admissible ordinal $\alpha$. In respect to this direction, for instance, it may be important to study the distribution of the reals contained in a $\Delta^1_1$ or $\Pi^1_1$ strong measure zero set in the hyperdegrees.

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