

ON THE STRENGTH OF MARRIAGE THEOREMS AND UNIFORMITY

MAKOTO FUJIWARA, KOJIRO HIGUCHI, AND TAKAYUKI KIHARA

ABSTRACT. Kierstead showed that every computable marriage problem has a computable matching under the assumption of computable expanding Hall condition and computable local finiteness for boys and girls. The strength of the marriage theorem reaches WKL_0 or ACA_0 if computable expanding Hall condition or computable local finiteness for girls is weakened. In contrast, the provability of the marriage theorem is maintained in RCA even if local finiteness for boys is completely removed. Using these conditions, we classify the strength of variants of marriage theorems in the context of reverse mathematics. Furthermore, we introduce another condition that also makes the marriage theorem provable in RCA_0 , and investigate the sequential and Weihrauch strength of marriage theorems under that condition.

1. INTRODUCTION

1.1. **Summary.** A subset R of the product $B \times G$ of two sets can be thought of as a multi-valued function (a set-valued function, or a bipartite graph), and written as $R : B \rightrightarrows G$. Given a multi-valued function R between countable sets B and G , we discuss whether it has a single-valued *injective* selection or not. Such a problem is called a *marriage problem*. Hall [11] showed that the marriage problem for a multi-valued function $R : B \rightrightarrows G$ is true whenever R is locally finite (i.e., $R(b)$ has at most finitely many values for every $b \in B$) and fulfills the Hall condition (i.e., the cardinality of $R[X]$ is not less than that of X for every finite set $X \subseteq B$). However, in the early age of recursive graph theory (cf. [9]), Manaster and Rosenstein [18] found that such a multi-valued function need not have a single-valued *computable* injective selection, even if it has a *computable* graph and its local finiteness is *computably* confirmed. To render the marriage theorem *computable*, Kierstead [17] introduced the notion of expanding Hall condition, which indicates that the difference between $|R[X]|$ and $|X|$ tends to infinity as $|X|$ tends to infinity, where X ranges over all finite subsets of B . Then, he found that R has a single-valued *computable* injective selection whenever the graph of R is *computable*, the local finiteness of R is *computably* confirmed and the expanding Hall condition for R is *computably* witnessed.

Concepts such as local finiteness and the Hall condition can be thought of as width conditions for Π_1^0 classes, because the set of all injective selections of a multi-valued function forms a Π_1^0 class in $G^B \simeq \omega^\omega$ ([4]). For instance, the local finiteness is known simply as compactness, and its computable version is known as recursive boundedness. Based on this observation, in the context of reverse mathematics, Hirst [13] (see also [12]) showed that the finite marriage theorem is provable in RCA_0 , and that the infinite marriage theorem is equivalent to ACA (equivalently, König's lemma) over RCA_0 . Moreover, he showed that the infinite marriage theorem under the assumption of computable local finiteness is equivalent to WKL (König's lemma for binary trees) over RCA_0 .

Our aim is to clarify the relationship between such width conditions for Π_1^0 classes (problems) and the complexity of elements contained in them (solutions to them) in the context of reverse mathematics. To achieve this, in section 2 we investigate the strength of 24 variations (except 3 false variations) of marriage theorems obtained by combining three *smallness* conditions (namely, no local finiteness (B, G), local finiteness (B', G'), and highly recursiveness (B'', G'')) and three *largeness* conditions (namely, Hall condition (H), expanding Hall condition (H'), and computable expanding Hall condition (H'')).

The computable expanding Hall condition (H'') guarantees that every large number of inputs has sufficiently large number of outputs in order to make a computable marriage problem have a computable solution; however, there is another condition that implies this. In section 3, we introduce a new condition called *constant bounded Hall condition* (H_{cb}), which requires that every finite set of inputs has few extra outputs. If a computable marriage problem fulfills this condition, then this problem will have a “non-uniformly” computable solution. In the practice of reverse mathematics, the sequential versions of Π_2^1 theorems, which expects to solve infinitely many instances of a particular problem simultaneously, have been investigated in order to reveal the necessity of the non-uniformity of their proofs in RCA_0 . For instance, the intermediate value theorem is provable in RCA_0 but its sequential version is equivalent to WKL [19]. We show in Section 3 that all of the marriage theorems with the constant bounded Hall condition (H_{cb}) are “non-uniformly” provable in RCA_0 , while some of their sequential versions are equivalent to WKL or ACA over RCA_0 .

The sequential strength roughly suggests the non-uniformity level of computable principles. However, even if a Π_2^1 theorem τ is provable in RCA and its sequential version $Seq(\tau)$ is equivalent to WKL , it is considerably short of

determining the exact computational strength of τ . In Section 4, we employ the notion of Weihrauch reducibility to further analyze the computational strength of marriage theorems. Our investigation demonstrates the close connection between sequential reverse mathematics [14, 7, 6] and Brattka-Gherardi style reverse mathematics [2, 3] via Weihrauch reducibility.

The reader is referred to Simpson's book [19] for basic knowledge of reverse mathematics including techniques for encoding mathematical statements in second order arithmetic. See [10] for the basic discussions on cardinality in weak first order arithmetic and the first order hierarchy.

1.2. Basic Terminology and Notation. Throughout this paper, for sets B and G , we often identify each bipartite graph $R(B, G)$ with a multi-valued function $R \subseteq B \times G$ defined by $R(b) = \{g \in G : (b, g) \in R\}$. In addition, when the underlying sets B and G of a bipartite graph $R(B, G)$ are clear from the context, we drop B, G and denote the bipartite graph just as R to avoid the notational complexity. In the graph theoretic terminology, each element of B and G is called "vertex", and each element (b, g) of R is called "edge". One can think of B and G as the set of boys and girls, respectively. Then $(b, g) \in R$ is regarded as that boy b knowing girl g .

For a function $f : B \rightarrow G$ and sets $X \subseteq B$ and $Y \subseteq G$, let $f[X]$ and $f^{-1}[Y]$ be the image of X and the preimage of Y under f , and $\text{dom}(f)$ and $\text{rng}(f)$ be the domain and the range of f . We also use the same notation for a multi-valued function $R \subseteq B \times G$. That is, $R[X]$ and $R^{-1}[Y]$ denote $\{g \in G : (\exists b \in X) g \in R(b)\}$ and $\{b \in B : (\exists g \in Y) g \in R(b)\}$ respectively. In addition, by $\text{dom}(R)$ and $\text{rng}(R)$ we mean $\{b \in B : R(b) \neq \emptyset\}$ and $R[B]$ respectively. Given $Z \subseteq R$, by $R - Z$ we denote the bipartite graph (multi-valued function) with $\text{dom}(R - Z) = \text{dom}(R) \setminus \text{dom}(Z)$ and $(R - Z)(b) := R(b) \setminus \text{rng}(Z)$ for every $b \in \text{dom}(R - Z)$. Given $V \subseteq B \cup G$, by $R - V$ we denote the bipartite graph with $\text{dom}(R - V) = \text{dom}(R) \setminus (V \cap B)$ defined by $(R - V)(b) = R(b) \setminus (V \cap G)$ for every $b \in \text{dom}(R - V)$. We denote a sequence by $\langle \cdot \rangle$. For a (code of) sequence s , $lh(s)$ denotes the length of s , s_i denotes the i -th element of s for $i < lh(s)$, and $s \hat{\ } \langle t \rangle$ denotes the concatenation of s and $\langle t \rangle$. C^* is the set of all finite sequences of the elements of a set C . For a given set B , $X \subseteq_{\text{fin}} B$ denotes that X is a (code of) finite subset of B . Let $S_R(X) := |R[X]| - |X|$ for $X \subseteq_{\text{fin}} B$.

We recall that RCA_0 consists of basic axioms for arithmetic, the Σ_1^0 induction scheme and the Δ_1^0 comprehension scheme, WKL_0 consists of RCA_0 and WKL (weak König's lemma), and ACA_0 consists of RCA_0 and ACA (the arithmetical comprehension scheme). (See [19] for the formal definitions.) In addition, $\Sigma_n^0\text{-IND}$ denotes the Σ_n^0 induction scheme and RCA denotes the extension of RCA_0 with the full second order induction scheme.

1.3. Main Results. First, we state the precise definition of each notion in RCA_0 . A bipartite graph $R(B, G)$ is *B-locally finite* if $|R[b]| < \infty$ for all $b \in B$. The graph is *B-highly recursive* or *computably B-locally finite* if there is a function $f : B \rightarrow \mathbb{N}$ such that $f(b) = |R[b]|$ for all $b \in B$. The notion of being *G-local finiteness* and *G-highly recursiveness* are defined in the same manner. The graph $R(B, G)$ satisfies *the Hall condition* if $|R[X]| \geq |X|$ holds for all $X \subseteq_{\text{fin}} B$; it satisfies *the expanding Hall condition* if it satisfies the Hall condition, and, for every $n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that the difference $S_R(X) = |R[X]| - |X|$ is not less than n for all $X \subseteq_{\text{fin}} B$ such that $|X| \geq m$. If there is a function in RCA_0 mapping each n to such an m , then we say that $R(B, G)$ satisfies *the computable expanding Hall condition*. A *solution*, a *matching* or an *injective selection* of $R(B, G)$ is an injective single-valued function $M \subseteq R$, i.e., an injection $M : B \rightarrow G$ with $M(b) \in R(b)$. Hereafter, we use the following notation:

- X : no local finiteness for X , for $X \in \{B, G\}$.
- X' : X -locally finite, for $X \in \{B, G\}$.
- X'' : X -highly recursive, for $X \in \{B, G\}$.
- H : Hall condition.
- H' : expanding Hall condition.
- H'' : computable expanding Hall condition.

Statement $(B_{H^{(\cdot)}}^{(\cdot)} G^{(\cdot)}\text{-M})$. If a bipartite graph $R(B, G)$ satisfies $B^{(\cdot)}$, $G^{(\cdot)}$, and $H^{(\cdot)}$, then $R(B, G)$ has a solution.

We investigate the strength of all possible marriage theorems having the above form. Using this terminology, we can rephrase the result of Hirst [13] as follows: The statement $B_H' G\text{-M}$ is equivalent to ACA , and $B_H'' G\text{-M}$ is equivalent to WKL over RCA_0 . Furthermore, Kierstead [17] showed that $B_{H''}'' G''\text{-M}$ holds effectively. Now we note that the Hall condition has Π_2^0 form but it can be written as a Π_1^0 formula under the assumption of being B -highly recursive. Thus, we can verify Kierstead's proof in our base system RCA_0 with the Σ_1^0 induction scheme (which enable us to carry out Π_1^0 induction [19, Corollary II.3.10]). Hence, $B_{H''}'' G''\text{-M}$ is provable in RCA_0 . Furthermore, the sequential version of $B_{H''}'' G''\text{-M}$ is also provable in RCA_0 by imitating the proof.

Our results in Section 2 are summarized in Table 1. Consequently, we find that the two conditions G'' and H'' are necessary and sufficient for a computable marriage problem to have a computable solution. Except in these cases, $G^{(\cdot)}$ and $H^{(\cdot)}$ do not affect the strength of marriage theorems, and only condition B'' is necessary and sufficient for a marriage theorem to be provable in WKL_0 .

	Hall condition	Expanding Hall condition	Recursive expanding Hall condition
ACA ₀	*	B _{H'} G-M	B _{H''} G-M
	*	B _{H'} G'-M	B _{H''} G'-M
	*	B _{H'} G''-M	B _{H''} G''-M
	B' _H G-M [13]	B' _H G-M	B' _H G-M
	B' _H G'-M	B' _H G'-M	B' _H G'-M
	B' _H G''-M	B' _H G''-M	B' _H G''-M
WKL ₀	B'' _H G-M [13]	B'' _H G-M	B'' _H G-M
	B'' _H G'-M	B'' _H G'-M	B'' _H G'-M
	B'' _H G''-M	B'' _H G''-M	B'' _H G''-M

TABLE 1. The strength of marriage theorems (* : false)

ACA ₀	Seq(B' _{H_{cb}} G-M)	Seq(B'' _{H_{cb}} G'-M)	Seq(B'' _{H_{cb}} G''-M)
WKL ₀	Seq(B'' _{H_{cb}} G-M)	Seq(B'' _{H_{cb}} G'-M)	Seq(B'' _{H_{cb}} G''-M)
RCA ₀			

TABLE 2. The sequential strength of constant bounded marriage theorems, which are provable in RCA₀

On the other hand, already mentioned in Subsection 1.1, there is another approach to make a computable marriage problem have a computable solution. As an extreme case, if the Hall condition holds and each boy knows at most one girl, then it obviously has a computable solution. Based on this observation, we introduce another kind of Hall condition, which requires that boys have few extra acquaintances. A bipartite graph $R(B, G)$ satisfies *the constant bounded Hall condition* if there exists k such that for all $X \subset_{\text{fin}} B$, $|X| \leq |R[X]| \leq |X| + k$ holds. We use the symbol H_{cb} for the constant bounded Hall condition. Our results in Section 3 are summarized in Table 2. Summarizing, a marriage theorem with condition H_{cb} is uniformly computable if and only if it includes G'' .

2. MARRIAGE THEOREMS WITH EXPANDING AND RECURSIVE EXPANDING HALL CONDITION

2.1. Reversals. Kierstead [17, Theorem 5] showed that $B''_{H'}G''\text{-M}$ does not hold in the least ω -model of RCA₀, while Hirst [13, Theorem 2.3] showed that it is provable in WKL₀. The next lemma means that it is actually equivalent to WKL over RCA₀.

Lemma 2.1. $\text{RCA}_0 \vdash B''_{H'}G''\text{-M} \rightarrow \text{WKL}$, that is, the following assertion implies WKL over RCA₀: If $R(B, G)$ is a bipartite graph which is B, G -highly recursive and satisfies the expanding Hall condition, then $R(B, G)$ has a solution.

Proof. We extend Kierstead's proof of [17, Theorem 5] to show this lemma. We reason in RCA₀. It suffices to separate the range of disjoint injections, which is equivalent to WKL over RCA₀ ([19, Lemma IV.4.4]). Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be given injections with pairwise disjoint ranges. The basic idea of the proof is to construct infinitely many disjoint marriage problems such that the solution of the i -th problem indicates whether i is in $\text{rng}(f)$ or $\text{rng}(g)$. That is, the bipartite graph $R(B, G)$ produced eventually is the disjoint union of the bipartite graphs $R_i(B_i, G_i)$, $i \in \mathbb{N}$. Here we describe the construction of the i -th graph $R_i(B_i, G_i)$. The underlying set of the graph $R_i(B_i, G_i)$ is the disjoint union of $3(i+1)$ many infinite full binary trees $\{0, 1\}^*$. Thus, each $v \in B_i \cup G_i$ is described as $(k, \sigma) \in T_i = 3(i+1) \times \{0, 1\}^*$. The sets of the boys B_i and the girls G_i in the graph $R_i(B_i, G_i)$ are chosen as $B_i = \{(k, \sigma) \in T_i : lh(\sigma) \text{ is even}\}$ and $G_i = \{(k, \sigma) \in T_i : lh(\sigma) \text{ is odd}\}$.

We construct a set R_i of edges on the graph $R_i(B_i, G_i)$ as follows (see also Fig. 1). At the first step, we in advance connect each boy in the 0-th column with his two successor girls in the 1-st column. Now we consider three cases for the construction of $R_i(B_i, G_i)$ in the $(2j+2)$ -th column. Let U_i^j be the set of the first $(i+1)2^{2j+1}$ boys in the $(2j+2)$ -th column, and let $G_i^{L(j)}$ (resp. $G_i^{R(j)}$) be the set of all $(k, \sigma) \in T_i$ with $\sigma(0) = 0$ (resp. $\sigma(0) = 1$) in the $(2j+1)$ -th column.

- (1) If neither $f(j) = i$ nor $g(j) = i$ we connect each boy in the $(2j+2)$ -th column with the predecessor girl in the $(2j+1)$ -th column and the two successor girls in the $(2j+3)$ -th column respectively.
- (2) If $f(j) = i$, then we connect U_i^j with $G_i^{L(j)}$ completely and each remaining boy in the $(2j+2)$ -th column with the two successor girls in the $(2j+3)$ -th column respectively.
- (3) If $g(j) = i$, then we connect U_i^j with $G_i^{R(j)}$ completely and each remaining boy in the $(2j+2)$ -th column with the two successor girls in the $(2j+3)$ -th column respectively.

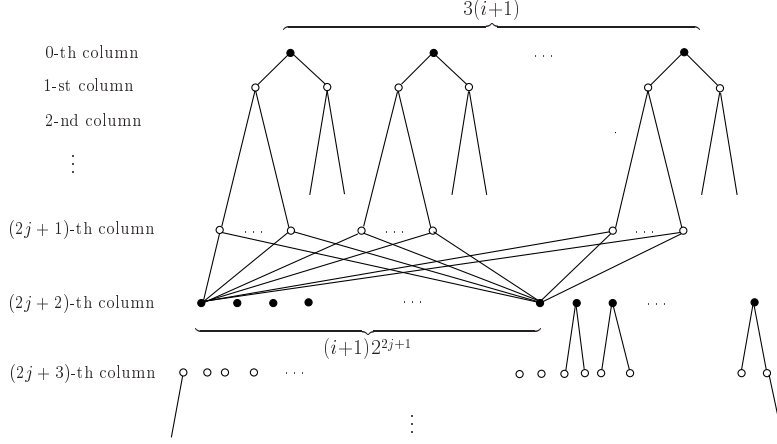


FIGURE 1. i -th graph $R_i(B_i, G_i)$ in the case of $f(j) = i$

Importantly, it inductively follows from this construction that for all $l \leq j$, at least the two-third of boys in the left (resp. right) sides in $2l$ -th column (see Fig. 1) choose their predecessor girls whenever $f(j) = i$ (resp. $g(j) = i$).

It is trivial that $R(B, G)$ is B, G -highly recursive. We show that $R(B, G)$ satisfies the expanding Hall condition.

Claim 2.2 (RCA₀). If i is contained in $\text{rng}(f) \cup \text{rng}(g)$, then either $S_{R_i}(X) \geq |X|$ or $S_{R_i}(X) \geq i + 1$ holds for $X \subset_{\text{fin}} B_i$.

(Proof of Claim.) Let i be contained in $\text{rng}(f) \cup \text{rng}(g)$ at j , and fix a finite subset X of B_i . If X and U_i^j are disjoint, then $S_{R_i}(X) \geq |X|$ holds since each boy in X is connected with the two successor girls. Assume that X intersects U_i^j . Consider the following set.

$$V_i^j = \begin{cases} \{(k, \sigma) \in B_i : 0 < lh(\sigma) \leq 2j, \text{ and } \sigma(0) = 0\} & \text{if } i \in \text{rng}(f), \\ \{(k, \sigma) \in B_i : 0 < lh(\sigma) \leq 2j, \text{ and } \sigma(0) = 1\} & \text{if } i \in \text{rng}(g). \end{cases}$$

Note that the boys in the 0-th column are not contained in V_i^j . We separate $R_i(B_i, G_i)$ into two subgraphs. Let $W_i := U_i^j \cup V_i^j$. Then $S_{R_i - W_i}(X \setminus W_i) \geq 0$ since each boy in $X \setminus W_i$ is connected with at least one successor girl. To estimate the value $S_{R_i}(X \cap W_i)$, we first note that this value is equal to $|R_i[X \cap W_i]| - |X \cap V_i^j| - |X \cap U_i^j|$. Moreover, $|R_i[X \cap W_i]| \geq |R_i[U_i^j]| + 2^{-1}|X \cap V_i^j|$ holds since each boy in U_i^j knows all girls in $R_i[U_i^j]$, and, for each boy in V_i^j , his predecessor girl has just two successor boys. By our definition, $|R_i[U_i^j]| = |G_i^{L(j)}| = |G_i^{R(j)}| = 3(i+1)2^{2j}$, $|U_i^j| = (i+1)2^{2j+1}$, and

$$|V_i^j| = 3(i+1) \sum_{l=0}^{j-1} 2^{2l+1} = 2(i+1)(2^{2j} - 1).$$

Therefore,

$$\begin{aligned} S_{R_i}(X \cap W_i) &\geq |R_i[U_i^j]| - \frac{1}{2}|X \cap V_i^j| - |X \cap U_i^j| \\ &\geq 3(i+1)2^{2j} - (i+1)(2^{2j} - 1) - (i+1)2^{2j+1} \\ &= i + 1. \end{aligned}$$

Hence, we have $S_{R_i}(X) \geq S_{R_i - W_i}(X \setminus W_i) + S_{R_i}(X \cap W_i) \geq i + 1$. \square

Let $W_{<n}$ be the union of the sets W_i such that $i < n$ and $i \in \text{rng}(f) \cup \text{rng}(g)$. If i is not contained in $\text{rng}(f) \cup \text{rng}(g)$, then $S_{R_i}(X) \geq |X|$ for $X \subset_{\text{fin}} B_i$, since the graph structure of $R_i(B_i, G_i)$ also corresponds exactly to the disjoint union of $3(i+1)$ many full binary trees. By this fact and Claim 2.2, it is not hard to see that for all n , if a finite subset X of B contains at least $n + |W_{<n}|$ elements, then $S_R(X) \geq n$ holds (see the proof of [17, Theorem 5] for details). Thus, $R(B, G)$ satisfies the expanding Hall condition.

Consequently, the assertion $B_{\text{H}}^{\text{M}}, G''\text{-M}$ ensures that $R(B, G)$ has a solution M . Let $M_{i,0}$ be the set of all boys b in the 0-th column of $R_i(B_i, G_i)$ who chooses the left successor girl according to M (i.e., $M(b) = (k, \langle 0 \rangle)$ for $k < 3(i+1)$). By Σ_0^0 comprehension, the set $S = \{i \in \mathbb{N} : |M_{i,0}| \leq i + 1\}$ exists. We shall show that S separates the

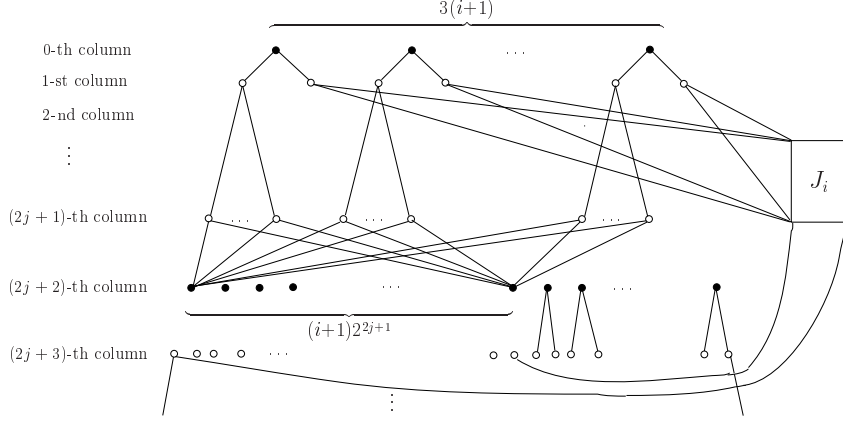


FIGURE 2. i -th graph $R_i(B_i, G_i)$ in the case of $f(j) = i$

ranges of f and g . Suppose $i \in \text{rng}(f)$ and $i \notin S$. Then $|M_{i,0}| > i + 1$. Now

$$|U_i^j \cup V_i^j| = |U_i^j| + |V_i^j| = (i+1)2^{2j+1} + 2(i+1)(2^{2j} - 1) = (i+1)2^{2j+2} - 2(i+1).$$

However,

$$|M[U_i^j \cup V_i^j]| = 3(i+1) \sum_{l=0}^j 2^{2l} - |M_{i,0}| < (i+1)2^{2j+2} - 2(i+1).$$

This contradicts the injectivity of M . Thus $i \in \text{rng}(f)$ implies $i \in S$. In the same manner, we can show that $i \in \text{rng}(g)$ implies $i \notin S$. \square

By extending the previous proof, we can show the next lemma.

Lemma 2.3. $\text{RCA}_0 \vdash B'_{\text{H}}G''\text{-M} \rightarrow \text{ACA}$, that is, the following assertion implies ACA over RCA_0 : If $R(B, G)$ is a bipartite graph which is B -locally finite, G -highly recursive and satisfies the expanding Hall condition, then $R(B, G)$ has a solution.

Proof. We reason in RCA_0 . It suffices to find the range of an injection, which is equivalent to ACA over RCA_0 ([19, Lemma III.1.3]). Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a given injection. We can show the existence of range of f in a slightly little different way from Lemma 2.1. As in the previous proof, we construct infinitely many disjoint marriage problems such that the solution of i -th problem indicates whether i is in $\text{rng}(f)$ or not. The construction of $R_i(B_i, G_i)$ is similar to the previous one. The underlying set of $R_i(B_i, G_i)$ consists of the disjoint union of $3(i+1)$ infinite binary trees pruned below the right girls in the 1-st column and with additional $2(i+1)$ boys. Formally, the underlying set $T_i = B_i \cup G_i$ of $R_i(B_i, G_i)$ is defined as $T_i = \{(k, \sigma) \in 3(i+1) \times \{0, 1\}^* : \sigma(0) \neq 1\} \cup J_i$, where J_i is a set of $2(i+1)$ vertices (see also Fig. 2), and put $B_i = \{(k, \sigma) \in T_i : \text{lh}(\sigma) \text{ is even}\} \cup J_i$ and $G_i = \{(k, \sigma) \in T_i : \text{lh}(\sigma) \text{ is odd}\}$.

We construct R_i as follows. In advance, we connect each boy in the 0-th column with his two successor girls in the 1-st column as before. Moreover, connect each right girl in the 1-st column with the exceptional boys in J_i completely. Note that the number of the right girls in the 1-st column is $3(i+1)$ and $|J_i| = 2(i+1)$. Then we determine who are connected with the boys in the $(2j+2)$ -th column according to f as follows. (See also Fig. 2.)

- (1) If $f(j) \neq i$, we connect each boy in the $(2j+2)$ -th column with the girls in the $(2j+1)$ -th column and $(2j+3)$ -th column as before.
- (2) if $f(j) = i$, we not only connect U_i^j with $G_i^{L(j)}$ completely and each remaining boy in the $(2j+2)$ -th column with the two successor girls in the $(2j+3)$ -th column respectively as before, but also connect each boy in J_i with some two girls remaining in the $(2j+3)$ -th column disjointly.

In the above construction, the procedure that combining J_i with the girls remaining in the $(2j+3)$ -th column has the role of “liberating” the right girls in the 1-st column from the proposal by the boys in J_i . Since we use this way of revising several times in the proofs below, we shall call this technique “*liberation method*”.

Obviously the graph $R(B, G)$ produced in this way is B -locally finite and G -highly recursive and we can show that $R(B, G)$ satisfies the expanding Hall condition as in Lemma 2.1. Then, the assertion $B'_{\text{H}}G''\text{-M}$ ensures that $R(B, G)$ has a solution M . As in the previous proof, let S be the set of all numbers i with $|M_{i,0}| \leq i + 1$, where $M_{i,0}$ is the set of boys in the 0-th column of $R_i(B_i, G_i)$ who choose their left successor girls according to M . As before, it is easy to see that $i \in \text{rng}(f)$ implies $i \in S$. If $i \notin \text{rng}(f)$, then the $2(i+1)$ boys in J_i just know the right girls in the 1-st

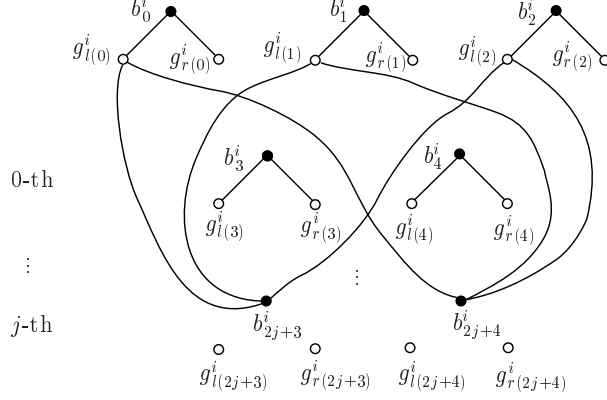


FIGURE 3. i -th graph $R_i(B_i, G_i)$ in the case of $f(j) = i$

column (i.e., the girls of form $(k, \langle 1 \rangle) \in T_i$). Therefore, $2(i+1)$ right girls in the 1-st column must be chosen by boys in J_i . Then, at most $i+1$ boys in the 0-th column can choose their right girls. Thus, $|M_{i,0}| > i+1$, and so $i \notin S$. Consequently, $S = \text{rng}(f)$. \square

Lemma 2.4. $\text{RCA}_0 \vdash \text{B}_{\text{H}''}''G'-\text{M} \rightarrow \text{WKL}$, that is, the following assertion implies WKL over RCA_0 : If $R(B, G)$ is a bipartite graph which is B -highly recursive, G -locally finite, and satisfies the computable expanding Hall condition, then $R(B, G)$ has a solution.

Proof. We reason in RCA_0 . It suffices to separate the range of disjoint functions ([19, Lemma IV.4.4]). Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be given injections with pairwise disjoint ranges. We construct a bipartite graph $R(B, G)$ which is the disjoint union of bipartite graphs $R_i(B_i, G_i)$, $i \in \mathbb{N}$. Put $B_i = G_i = \mathbb{N}$. For convenience, we will suppress the coding and label the j -th boy in B_i by b_j^i as well as the $2j$ -th and $(2j+1)$ -th girls in G_i by $g_{l(j)}^i$ and $g_{r(j)}^i$ respectively. Then we construct R_i as follows (see also Fig. 3).

- (1) The pairs $(b_u^i, g_{l(u)}^i)$ and $(b_u^i, g_{r(u)}^i)$ are enumerated into R_i for each $u \in \{0, 1, 2\}$.
- (2) If neither $f(j) = i$ nor $g(j) = i$ holds, then the pairs $(b_{2j+3+v}^i, g_{l(2j+3+v)}^i)$ and $(b_{2j+3+v}^i, g_{r(2j+3+v)}^i)$ are enumerated into R_i for each $v \in \{0, 1\}$.
- (3) If $f(j) = i$ holds, then $(b_{2j+3+v}^i, g_{l(u)}^i)$ is enumerated into R_i for each $u \in \{0, 1, 2\}$ and $v \in \{0, 1\}$.
- (4) If $g(j) = i$ holds, then $(b_{2j+3+v}^i, g_{r(u)}^i)$ is enumerated into R_i for each $u \in \{0, 1, 2\}$ and $v \in \{0, 1\}$.

It is trivial that $R(B, G)$ is B -highly recursive and G -locally finite. We show that $R(B, G)$ satisfies the computable expanding Hall condition.

Claim 2.5 (RCA_0). For a finite subset X of B , $|X| \geq 5n$ implies $S_R(X) \geq n$.

(Proof of Claim.) Let X be a finite subset of B such that $|X| \geq 5n$. For each i , let B'_i be the set of all boys of the form b_j^i such that $(b_j^i, g_{l(0)}^i)$ or $(b_j^i, g_{r(0)}^i)$ is enumerated into R . Since the range of f and g are disjoint, $S_R(X \cap B_i) \geq S_R(X \cap B'_i)$ for each $i \in \mathbb{N}$. Then $S_R(X) = \sum_i S_R(X \cap B_i) \geq \sum_i S_R(X \cap B'_i)$. Note that, if $X \cap B'_i \neq \emptyset$, then $S_R(X \cap B'_i) \geq 1$. Therefore, in the case of $|\{i : X \cap B'_i \neq \emptyset\}| \geq n$, we have $S_R(X) \geq n$. In the case of $|\{i : X \cap B'_i \neq \emptyset\}| \leq n$, there are at least $3n$ boys who are not in $\bigcup_i B'_i$. Since each of these boys knows under 2 girls (see Fig. 3), $|R[X]| \geq 6n$, so $S_R(X) = |R[X]| - |X| \geq 6n - 5n = n$. \square

By the previous claim, $R(B, G)$ satisfies the computable expanding Hall condition.

Then there exists a solution M of $R(B, G)$ by $\text{B}_{\text{H}''}''G'-\text{M}$. By Δ_1^0 comprehension, take

$$V := \left\{ i \mid \text{two of } \{b_0^i, b_1^i, b_2^i\} \text{ choose their right girls via } M \right\}.$$

If $i \in \text{rng}(f)$, then b_{2j+3}^i and b_{2j+4}^i chooses two of $\{g_{l(0)}^i, g_{l(1)}^i, g_{l(2)}^i\}$ via M , then two of $\{b_0^i, b_1^i, b_2^i\}$ must choose their right girls. Hence, $i \in V$. If $i \in \text{rng}(g)$, then b_{2j+3}^i and b_{2j+4}^i choose two of $\{g_{r(0)}^i, g_{r(1)}^i, g_{r(2)}^i\}$ via M . Hence, $i \in V$. \square

In the context of recursive graph theory, Lemma 2.4 suggests that being G -highly recursive is essential for a computable marriage problem with the computable expanding Hall condition to have a computable solution.

By applying ‘‘liberation method’’ to the proof of Lemma 2.4, we can show the next lemma.

Lemma 2.6. $\text{RCA}_0 \vdash \text{B}'_{\text{H}''}G'$ -M \rightarrow ACA, that is, the following assertion implies ACA over RCA_0 : If $R(B, G)$ is a bipartite graph which is B, G -locally finite and satisfies the computable expanding Hall condition, then $R(B, G)$ has a solution.

Proof. We reason in RCA_0 . It suffices to find the range of an injection $f : \mathbb{N} \rightarrow \mathbb{N}$ ([19, Lemma III.1.3]). The construction of R is similar to that in Lemma 2.4. But in this occasion, for each i -th graph $R_i(B_i, G_i)$, we make the top right three girls being connected by two exceptional boys b_{el}^i, b_{er}^i in advance. Then if $f(j) = i$, we carry out the following procedure in the i -th graph $R_i(B_i, G_i)$.

- (1) Combine b_{2j+3}^i, b_{2j+4}^i with the top left girls g_0^i, g_1^i, g_2^i completely.
- (2) Combine b_{el}^i with $g_{l(2j+3)}^i, g_{r(2j+3)}^i$ and b_{er}^i with $g_{l(2j+4)}^i, g_{r(2j+4)}^i$.

The procedure 2 has the role of “liberating” the top right three girls from the proposal by b_{el}^i, b_{er}^i . Then one can see that $R(B, G)$ is B, G -locally finite and satisfies the computable expanding Hall condition by taking $h : \mathbb{N} \rightarrow \mathbb{N}$ such that $h(n) = 5n$ as in Lemma 2.4. $\text{B}'_{\text{H}''}G'$ -M ensures that $R(B, G)$ has a solution M and $V := \{i \mid \text{two of } \{b_0^i, b_1^i, b_2^i\} \text{ choose their right girls via } M\}$ is the range of f as before. \square

2.2. Proofs in $\text{RCA}_0 + \Sigma_3^0$ -IND. We recall that a computable marriage problem has a computable solution under the three strongest assumptions $B'', G'',$ and H'' as shown in [17]. As we have seen in Lemmas 2.1 and 2.4, in general, it has *no* computable solution under the absence of G'' or H'' . What will happen when B'' is weakened? Surprisingly, even in the absence of B' , every computable marriage problem has a computable solution when it satisfies G'' and H'' .

Theorem 2.7. $\text{RCA}_0 + \Sigma_3^0$ -IND $\vdash \text{B}_{\text{H}''}G''$ -M, that is, the following is provable in $\text{RCA}_0 + \Sigma_3^0$ -IND: If $R(B, G)$ is a bipartite graph which is G -highly recursive and satisfies the computable expanding Hall condition, then $R(B, G)$ has a solution.

We need the following lemmas to show the previous theorem.

Lemma 2.8 ($\text{RCA}_0 + \Sigma_2^0$ -IND). Assume that $R(B, G)$ is a bipartite graph with the expanding Hall condition. Then for any $b \in B$ there exists $g \in R[b]$ such that the remaining graph $R - \{(b, g)\}$ satisfies the Hall condition.

Proof. We consider in two cases. We first consider in the case that given b knows infinitely many girls. If $|X| < |R[X]|$ for all $X \subseteq_{\text{fin}} B$, our requirement clearly holds. We assume that there exists $X \subseteq_{\text{fin}} B$ such that $|X| \geq |R[X]|$, which implies $|X| = |R[X]|$ by the Hall condition. Then there exists a maximal finite subset $X_0 \subseteq_{\text{fin}} B$ such that $|X_0| \geq |R[X_0]|$, since if it is not, Σ_2^0 -IND induction can prove a contradiction to the expanding Hall condition. (Note that $|X_0| \geq |R[X_0]|$ is written as a Π_1^0 formula.) Now $b \notin X_0$. By using Π_1^0 collection principle (provable in $\text{RCA}_0 + \Sigma_2^0$ -IND), $R[X_0]$ is assured to be finite. Therefore, b must know some $g \notin R[X_0]$. We shall show that $R - \{(b, g)\}$ satisfies the Hall condition. Suppose not. Then there exists nonempty $X_1 \subseteq_{\text{fin}} B$ such that $|X_1| = |R[X_1]|$ and $g \in R[X_1]$. So, $|X_0 \cup X_1| = |R[X_0] \cup R[X_1]|$ follows from it. Moreover $X_0 \cup X_1 \supsetneq X_0$ follows from $g \in R[X_1] \setminus R[X_0]$. This contradicts the maximality of X_0 .

Secondly, we consider in the case that given boy b knows at most finitely many girls. Let B_f denote the set of all boys knowing at most finitely many girls. When B_f is finite, $\text{RCA}_0 + \Sigma_2^0$ -IND proves the existence of B_f , since B_f is Σ_2^0 definable and Σ_2^0 -IND implies bounded Σ_2^0 comprehension (cf. [19, Exercise II.3.13]). Therefore, our requirement clearly holds because the finite marriage theorem is provable in RCA_0 ([13, Theorem 2.1]). Next we assume that B_f is infinite. For the sake of contradiction, suppose that there is no $g \in R[b]$ such that $R - (b, g)$ satisfies the Hall condition. Since the original graph $R(B, G)$ satisfies the Hall condition, there is a finite set $X_g \subseteq R^{-1}[g] \setminus \{b\}$ such that $|X_g| = |R[X_g]|$ for all $g \in R[b]$. Put $X_1 = \bigcup_{g \in R[b]} X_g$. Then, since $R[b]$ is finite, Σ_2^0 -IND implies that X_1 is finite and $|X_1| = |R[X_1]|$. Since $b \notin X_1$ and $R[b] \subseteq R[X_1]$, we have

$$|X \cup \{b\}| = |R[X_1]| + 1 > |R[X_1]| = |R[X_1 \cup \{b\}]|.$$

This is the desired contradiction. Hence, there is $g \in R[b]$ such that $R - (b, g)$ satisfies the Hall condition. \square

Definition 2.9 (R -chain, chainable matching).

- (1) A finite sequence $s = \langle s_j^B, s_j^G \rangle_{j < k}$ is a R -chain with starting point b of length k (> 0) if $\langle s_j^B \rangle_{j < k}$ and $\langle s_j^G \rangle_{j < k}$ are nondecreasing sequences of finite subsets of B and G respectively, where $s_0^B = \{b\}$, $s_j^G \subseteq R[s_j^B]$, $s_{j+1}^B = R^{-1}[s_j^G]$, and $R(s_j^B, s_j^G)$ satisfies the Hall condition for each $j < k$. A R -chain $\langle s_j^B, s_j^G \rangle_{j < k}$ is called *proper* if $\langle s_j^B \rangle_{j < k}$ is strictly increasing.
- (2) A finite matching $m \subseteq_{\text{fin}} R$ is R -chainable with starting point b if there is an R -chain $s = \langle s_j^B, s_j^G \rangle_{j < k}$ with starting point b such that $\text{dom}(m) = s_{k-1}^B$, and $s_j^G = m[s_j^B]$ for all $j < k$.

Note that every R -chainable matching m with an R -chain s has the following disjointness property:

$$(1) \quad R[B \setminus s_{j+1}^B] \cap m[s_j^B] = \emptyset,$$

for each $j < lh(s) - 1$. The next technical lemma has a key role in our proof of Theorem 2.7.

Lemma 2.10 ($\text{RCA}_0 + \Sigma_3^0\text{-IND}$). *Assume that $R(B, G)$ is a bipartite graph which is G -highly recursive and satisfies the expanding Hall condition. Then, for all $b \in B$ and $l \in \mathbb{N}$, there exists an R -chainable matching m with starting point b which has an R -chain of length l such that $R - m$ satisfies the Hall condition.*

Proof. Since $R(B, G)$ is G -highly recursive, there exists a function q from codes of finite subset Y of G to that of B such that $q(Y) = R^{-1}[Y]$. We fix $b \in B$ and show our lemma by induction on l . Since the Hall condition is a Π_2^0 formula, this is a Σ_3^0 induction and carried out straightforwardly by the iterative use of Lemma 2.8 and the function q within $\text{RCA}_0 + \Sigma_3^0\text{-IND}$. \square

We are now prepared to show Theorem 2.7.

PROOF OF THEOREM 2.7. We reason in $\text{RCA}_0 + \Sigma_3^0\text{-IND}$. It suffices to consider only the case that B is infinite since the finite marriage theorem is provable in RCA_0 ([13, Theorem 2.1]). Let $\{b_i : i \in \mathbb{N}\}$ be an enumeration of B and h be a witness of the computable expanding Hall condition. We shall now construct a solution of $R(B, G)$ by a recursive procedure. Let $\theta(u, v)$ say that $u = \langle u_i \rangle_{i \leq v}$ is a sequence of length $v + 1$ and for all $i < v + 1$, u_i is the least R_i -chainable matching with starting point b_i which has an R_i -chain of length $h(i + 1) + 1$, where $R_i(B_i, G_i) := R(B, G) - \{(b_{i'}, u_{i'}(b_{i'})) : i' < i\}$, namely, the remaining graph obtained by removing the previously determined i marriage pairs from $R(B, G)$. If $\theta(u, v)$ holds, then we identify each R_i -chainable matching u_i with the R_i -chain s_i for u_i .

Note that $\theta(u, v)$ is Σ_3^0 , since $R(B, G)$ is G -highly recursive. We shall decide the partner of b_i as the girl indicated via uniquely determined matching u_i . Suppose that we have shown $\forall v \exists u \theta(u, v)$. Then the witness u^v for each v is unique and u^{v_1} is an initial segment of u^{v_2} for $v_1 \leq v_2 \leq v$ because of the minimality of each u_i in the definition of $\theta(u, v)$. Therefore by Δ_1^0 comprehension (as primitive recursion in RCA_0 , see [19, Theorem II.3.4]), there exists a function which outputs the unique u^v for each $v \in \mathbb{N}$. Take $M : B \rightarrow G$ by $M(b_v) = (u^v)_v(b_v)$, then it is not hard to see that M is an injection from B to G . Thus our goal is to prove $\forall v \exists u \theta(u, v)$. In preparation, we first show the following claim.

Claim 2.11 ($\text{RCA}_0 + \Sigma_2^0\text{-IND}$). For all u and v , if $\theta(u, v)$ holds, then $R - \{(b_j, u_j(b_j)) : j < v + 1\}$ satisfies the Hall condition.

(Proof of Claim.) We fix u and v such that $\theta(u, v)$ holds and show that for all $i \leq v$, $R - \{(b_j, u_j(b_j)) : j < i + 1\}$ satisfies the Hall condition by induction on i . Since the Hall condition is Π_2^0 , this is a Π_2^0 induction. We shall show only the initial step below. The induction step can be shown in the same manner by using the induction hypothesis. Let $R_1 := R - \{(b_0, u_0(b_0))\}$, s_0 be an R -chain for u_0 and fix $X \subseteq_{\text{fin}} B \setminus \{b_0\}$.

In the case that $s_0 = \langle s_{0,j} \rangle_{j \leq h(1)}$ is non-proper. Let $\langle s_{0,j} \rangle_{j \leq k}$ be its least non-proper initial segment. By the disjointness property (1) with $s_{0,k-1}^B = s_{0,k}^B$,

$$\begin{aligned} |R_1[X]| &\geq |u_0[X \cap s_{0,k}^B] \cup R_1[X \setminus s_{0,k}^B]| \\ &= |u_0[X \cap s_{0,k}^B]| + |R_1[X \setminus s_{0,k}^B]| \\ &\geq |X \cap s_{0,k}^B| + |X \setminus s_{0,k}^B| = |X|, \end{aligned}$$

where the first inequality holds since $u_0 - \{(b_0, u_0(b_0))\} \subseteq R_1$ and the last inequality holds since u_0 is single-valued, R satisfies the Hall condition, and $R_1[X \setminus s_{0,k}^B] = R[X \setminus s_{0,k}^B] \setminus u_0[s_{0,0}^B] = R[X \setminus s_{0,k}^B]$ follows from the disjointness property (1).

Otherwise, i.e., $s_0 = \langle s_{0,j} \rangle_{j \leq h(1)}$ is proper. If $|X| \geq h(1)$, then $|R_1[X]| - |X| \geq |R[X]| - |X| - 1 \geq 0$ since the original graph $R(B, G)$ satisfies the expanding Hall condition via h . We consider in the case of $|X| < h(1)$. By properness, $a_j := s_{0,j}^B \setminus s_{0,j-1}^B$ is nonempty for each $j \leq h(1)$. Then, $a_{j_1} \cap X = \emptyset$ holds for some $0 < j_1 \leq h(1)$, since $\{a_j\}_{j \leq h(1)}$ is pairwise disjoint. Fix such j_1 . By the disjointness property (1) with the fact $X \cap s_{0,j_1}^B = X \cap s_{0,j_1-1}^B$, as in the previous paragraph,

$$\begin{aligned} |R_1[X]| &\geq |u_0[X \cap s_{0,j_1-1}^B] \cup R_1[X \setminus s_{0,j_1-1}^B]| \\ &= |u_0[X \cap s_{0,j_1-1}^B]| + |R_1[X \setminus s_{0,j_1-1}^B]| \\ &\geq |X \cap s_{0,j_1-1}^B| + |X \setminus s_{0,j_1-1}^B| = |X|, \end{aligned}$$

as desired. \square

We are now prepared to show by Σ_1^0 induction that $\forall v \exists u \theta(u, v)$ holds. For the initial step, it clearly holds by Lemma 2.10. We consider the induction step. Let $\theta(\hat{u}, v)$ hold. By Claim 2.11, the remaining graph $R - \{(b_j, (\hat{u})_j(b_j)) : j < v + 1\}$ satisfies the Hall condition. Define $\hat{h} : \mathbb{N} \rightarrow \mathbb{N}$ as

$$\begin{cases} \hat{h}(0) = 0, \\ \hat{h}(x) = h(x + v + 1) \quad \text{for } x > 0. \end{cases}$$

By Σ_0^0 comprehension, such \hat{h} exists. It is clearly a witness of the computable expanding Hall condition for our remaining graph. By Lemma 2.10, there exists an appropriate chainable matching \hat{m} , and it is easy to see that $\theta(\hat{u} \hat{\smallfrown} \langle \hat{m} \rangle, v + 1)$ holds. This completes the proof of our theorem. \square

As an immediate consequence from Theorem 2.7, we can see that $B'_{H''}G''$ -M is also provable in $\text{RCA}_0 + \Sigma_3^0\text{-IND}$, whereas we do not know whether $\Sigma_3^0\text{-IND}$ is necessary to prove $B_{H'}G'$ -M and $B'_{H''}G''$ -M. We also note that the construction of a solution in the proof of Theorem 2.7 is uniform. Hence, the sequential version of $B_{H'}G'$ -M is provable in $\text{RCA}_0 + \text{I}\Sigma_3^0$ by imitating the proof of Theorem 2.7.

2.3. Summary of Section 2.

Remark 2.12. As a conclusion from Lemma 2.1 and Lemma 2.4, and Theorem 2.7, it turns out that *the computable expanding Hall condition and being G -highly recursive are essential for a computable marriage problem to have a computable solution.*

Theorem 2.13. *All of the following assertions are equivalent to WKL over RCA_0 .*

$$\begin{array}{ccc} B''_H G\text{-M} & B''_{H'} G\text{-M} & B''_{H''} G\text{-M} \\ B''_H G'\text{-M} & B''_{H'} G'\text{-M} & B''_{H''} G'\text{-M} \\ B''_H G''\text{-M} & B''_{H'} G''\text{-M} & \end{array}$$

Proof. It is clear by the fact that $\text{WKL}_0 \vdash B''_H G\text{-M}$ [13, Theorem 2.3], Lemma 2.1 and Lemma 2.4 since the computable expanding Hall condition and being highly recursive are restrictions of the expanding Hall condition and being locally finite respectively. \square

Proposition 2.14. $\text{ACA}_0 \vdash B_{H'}G\text{-M}$, that is, the following is provable within ACA_0 : *If $R(B, G)$ is a bipartite graph which satisfies the expanding Hall condition, then $R(B, G)$ has a solution.*

Proof. Straightforward by a routine inspection of [17, Theorem 6]. \square

Theorem 2.15. *All of the following assertions are equivalent to ACA over RCA_0 .*

$$\begin{array}{ccccc} B'_H G\text{-M} & B_{H'} G\text{-M} & B'_{H'} G\text{-M} & B_{H''} G\text{-M} & B'_{H''} G\text{-M} \\ B'_H G'\text{-M} & B_{H'} G'\text{-M} & B'_{H'} G'\text{-M} & B_{H''} G'\text{-M} & B'_{H''} G'\text{-M} \\ B'_H G''\text{-M} & B_{H'} G''\text{-M} & B'_{H'} G''\text{-M} & & \end{array}$$

Proof. It is clear by the fact that $\text{ACA}_0 \vdash B'_H G\text{-M}$ [13, Theorem 2.2], Proposition 2.14, Lemma 2.3 and Lemma 2.6. \square

3. MARRIAGE THEOREMS WITH CONSTANT BOUNDED HALL CONDITION

3.1. Hall Condition with Constant Bound. In this section, we study reverse mathematics of marriage theorems with the constant bounded Hall condition H_{cb} which is introduced in Subsection 1.3. The constant bounded Hall condition means that there are very few choices of partners of each boy. Of course, it guarantees B -local finiteness B' . However, it does not help to make a computable marriage problem B -highly recursive. Nevertheless, the next theorem states that the constant bounded Hall condition renders marriage problems solvable in RCA_0 .

Theorem 3.1. $\text{RCA}_0 \vdash B'_{H_{cb}}G\text{-M}$, that is, the following is provable within RCA_0 : *If $R(B, G)$ is a bipartite graph which satisfies the constant bounded Hall condition, then $R(B, G)$ has a solution.*

Proof. We reason in RCA_0 . Let

$$\Phi(c) \equiv \forall X \subset_{\text{fin}} B (|R[X]| \leq |X| + c).$$

Note that the statement $|R[X]| \leq |X| + c$ is written as a Π_1^0 formula, and then so is $\Phi(c)$. Since $R(B, G)$ satisfies the constant bounded Hall condition, $\exists c \Phi(c)$ holds. By Π_1^0 least number principle, which can be carried out in RCA_0 , there exists a least c_1 such that $\Phi(c_1)$ holds. It follows from the leastness that there exists $X_1 \subset_{\text{fin}} B$ such that $|R[X_1]| = |X_1| + c_1$. We fix such X_1 . Then the set $R[X_1]$ exists by Σ_0^0 comprehension, and $|R[X_1]| < \infty$. We first note the following:

$$(2) \quad \text{For all } b \in B \setminus X_1, \text{ there is at most one } g \in R[b] \setminus R[X_1],$$

since if not, $|R[X_1 \cup \{b\}]| \geq |X_1 \cup \{b\}| + c_1 + 1$. Moreover, we claim that

$$X_2 := \{b \in B \setminus X_1 : R[b] \subseteq R[X_1]\}$$

is finite, hence exists by bounded Π_1^0 comprehension in RCA_0 (cf. [19, Theorem II.3.9]). Indeed, X_2 has at most c_1 many elements. Otherwise, for such a finite set X' of size $c_1 + 1$ with $R[X'] \subseteq R[X_1]$, we have $|X_1 \cup X'| \geq |X_1| + c_1 + 1 > |R[X_1]| = |R[X_1 \cup X']|$. Next, we claim that

$$Y_1 := \{g \in G \setminus R[X_1] : |R^{-1}[\{g\}] \setminus X_1| \geq 2\}$$

is finite (actually, of size at most c_1), and exists by bounded Σ_1^0 comprehension in RCA_0 [19, Theorem II.3.9]. Suppose not. Then there exists a finite set Y' of such girls that $|Y'| = c_1 + 1$. Moreover, by (2) with $Y' \cap R[X_1] = \emptyset$, for every different $g_1, g_2 \in Y'$, two sets $R^{-1}[g_1] \setminus X_1$ and $R^{-1}[g_2] \setminus X_2$ are disjoint. Then $R^{-1}[Y'] \geq 2|Y'| = 2(c_1 + 1)$ follows. Let X' be a finite subset of $R^{-1}[Y']$ such that $|X'| \geq 2(c_1 + 1)$. By (2), each boy in X' knows just one girl in Y' . Therefore,

$$|R[X_1 \cup X']| \leq |R[X_1]| + |Y'| = (|X_1| + c_1) + (c_1 + 1) = |X_1| + 2c_1 + 1.$$

On the other hand,

$$|X_1 \cup X'| = |X_1| + |X'| \geq |X_1| + 2c_1 + 2.$$

These contradict the Hall condition.

Now note that the condition (2) implies $R[R^{-1}[Y_1]] \subseteq R[X_1] \cup Y_1$. Therefore, we have

$$\begin{aligned} |X_1| + |R^{-1}[Y_1]| &= |X_1 \cup R^{-1}[Y_1]| \leq |R[X_1 \cup R^{-1}[Y_1]]| \\ &\leq |R[X_1] \cup Y_1| = |R[X_1]| + |Y_1| \leq (|X_1| + c_1) + c_1 = |X_1| + 2c_1. \end{aligned}$$

Hence, $R^{-1}[Y_1]$ has at most $2c_1$ many elements, and $R^{-1}[Y_1]$ exists by Σ_0^0 comprehension. Moreover, $|X_1 \cup X_2 \cup R^{-1}[Y_1]|$ is finite. On the other hand, $R[X_1 \cup X_2 \cup R^{-1}[Y_1]] \subseteq R[X_1] \cup Y_1$ holds by our choice of X_1 , X_2 and Y_1 . Thus the following finite subgraph

$$(X_1 \cup X_2 \cup R^{-1}[Y_1], R[X_1] \cup Y_1; R)$$

of $R(B, G)$ satisfies the Hall condition because of the Hall condition for the original graph $R(B, G)$. Then it has a matching M by the finite marriage theorem in RCA_0 ([13, Theorem 2.1]). Again by (2), each boy $b \notin X_1 \cup X_2 \cup R^{-1}[Y_1]$ knows just one girl $g_b \notin R[X_1] \cup Y_1$. Moreover, for any such boys b and b' , if $b \neq b'$, then $g_b \neq g_{b'}$, since $b, b' \notin R^{-1}[Y_1]$. Therefore, $M \cup \{(b, g_b) : b \in B \setminus (X_1 \cup X_2 \cup R^{-1}[Y_1])\}$ is a solution of $R(B, G)$ in RCA_0 . This completes the proof of our theorem. \square

Consequently, all of $B'_{\text{Hcb}} G'$ -M, $B'_{\text{Hcb}} G''$ -M, $B''_{\text{Hcb}} G$ -M, $B''_{\text{Hcb}} G'$ -M and $B''_{\text{Hcb}} G''$ -M are probable in RCA_0 . As a corollary, if a computable bipartite graph $R(B, G)$ satisfies the constant bounded Hall condition, then $R(B, G)$ has a computable solution. However, note that the algorithm in the proof of Theorem 3.1 to give a solution for a given instance of $B'_{\text{Hcb}} G$ -M is *not* uniform, in contrast to the uniformity of the algorithm in the proof of Theorem 2.7 for $B''_{\text{Hcb}} G''$ -M.

3.2. Sequential Reverse Mathematics.

3.2.1. *Extracting Non-uniformity from Proofs.* Our proof of Theorem 3.1 in RCA_0 contains an implicit non-uniformity in the use of least number principle. The next theorem suggests that this non-uniformity can not be avoided. We use a notation $\text{Seq}(A)$ for the sequential version of A below.

Theorem 3.2. *The following are pairwise equivalent over RCA_0 .*

- (1) ACA.
- (2) $\text{Seq}(B'_{\text{Hcb}} G\text{-M})$, that is, for all sequence $\langle B_n, G_n, R_n, k_n \rangle_{n \in \mathbb{N}}$ such that $R_n(B_n, G_n)$ satisfies the constant bounded Hall condition via k_n , there exists a sequence $\langle M \rangle_{n \in \mathbb{N}}$ of the solutions.¹
- (3) $\text{Seq}(B'_{\text{Hcb}} G'\text{-M})$, that is, for all sequence $\langle B_n, G_n, R_n, k_n \rangle_{n \in \mathbb{N}}$ such that $R_n(B_n, G_n)$ is G_n -locally finite and satisfies the constant bounded Hall condition via k_n , there exists a sequence $\langle M \rangle_{n \in \mathbb{N}}$ of the solutions.

Before proving this theorem, will we show the following related equivalences with WKL.

Theorem 3.3. *The following are pairwise equivalent over RCA_0 .*

- (1) WKL.
- (2) $\text{Seq}(B''_{\text{Hcb}} G\text{-M})$, that is, for all sequence $\langle B_n, G_n, R_n, p_n, k_n \rangle_{n \in \mathbb{N}}$ such that $R_n(B_n, G_n)$ is B_n -highly recursive via p_n and satisfies the constant bounded Hall condition via k_n , there exists a sequence $\langle M \rangle_{n \in \mathbb{N}}$ of the solutions.²

¹Note that the sequence k_n is included in the sequence of marriage problems. This is appropriate sequentialization because our focus is on the non-uniformity of the construction of a solution from the given constant bounded Hall condition via k . (cf. [8])

²Note that the sequence p_n is included in the sequence of marriage problems. This is the appropriate sequentialization for our purpose as in the previous footnote. (cf. [8])

- (3) $\text{Seq}(B''_{\text{Hcb}}, G' \text{-M})$, that is, for all sequence $\langle B_n, G_n, R_n, p_n, k_n \rangle_{n \in \mathbb{N}}$ such that $R_n(B_n, G_n)$ is B_n -highly recursive via p_n , G_n -locally finite and satisfies the constant bounded Hall condition via k_n , there exists a sequence $\langle M \rangle_{n \in \mathbb{N}}$ of the solutions.

Proof. (1 \rightarrow 2) holds by the facts that $\text{WKL}_0 \vdash B''_{\text{Hcb}} \text{G-M}$ ([13, Theorem 2.3]) and that $\text{RCA}_0 \vdash \text{WKL} \leftrightarrow \text{Seq}(\text{WKL})$ ([14, Lemma 5]). (2 \rightarrow 3) is trivial. We shall show (3 \rightarrow 1). It suffices to separate the ranges of disjoint functions ([19, Lemma IV.4.4]). Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be given injections with disjoint ranges.

We construct a sequence of bipartite graphs $\langle R_n(B_n, G_n) \rangle_{n \in \mathbb{N}}$ in RCA_0 . For each $n \in \mathbb{N}$, put $B_n = G_n = \mathbb{N}$. At first, (0, 0) and (0, 1) are enumerated into each R_n . At the j -th step in the construction of R_i , if $f(j) = i$ occurs, then put $(j + 1, 1) \in R_i$. If $g(j) = i$ occurs, then put $(j + 1, 0) \in R_i$. Otherwise, put $(j + 1, j + 2) \in R_i$.

We put $\langle p_n \rangle_{n \in \mathbb{N}} := \langle p \rangle_{n \in \mathbb{N}}$ where $p : \mathbb{N} \rightarrow \mathbb{N}$ such that $p(n) = n + 1$ and $\langle k_n \rangle_{n \in \mathbb{N}} := \langle 1 \rangle_{n \in \mathbb{N}}$ in RCA_0 . Then each n graph $R_n(B_n, G_n)$ is G_n -locally finite and B_n -highly recursive via p_n , and it is also easy to see that for all n and $X \subset_{\text{fin}} B_n$, $|X| \leq |R_n[X]| \leq |X| + k_n$ holds within RCA_0 . Then $\text{Seq}(B''_{\text{Hcb}}, G' \text{-M})$ implies the existence of a sequence $\langle M_i \rangle_{i \in \mathbb{N}}$ of solutions for $\langle R_n(B_n, G_n) \rangle_{n \in \mathbb{N}}$. Define $V := \{i : (0, 0) \in M_i\}$ by Σ_0^0 comprehension. Then V separates the ranges of f and g because of the above construction. \square

PROOF OF THEOREM 3.2. (1 \rightarrow 2) is shown straightforwardly by revising the proof of $\text{ACA}_0 \vdash B'_\text{H} \text{G-M}$ by Hirst ([13, Theorem 2.2]) a bit. (2 \rightarrow 3) is trivial. We show (3 \rightarrow 1) by revising a proof of (3 \rightarrow 1) of Theorem 3.3 by using “liberation method” as the proofs of Lemma 2.3 and Lemma 2.6.

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be an injection and for each $n \in \mathbb{N}$, put $B_n = G_n = \mathbb{N}$. At first, put $(0, 0), (0, 1), (1, 0) \in R_n$. At the j -th step in the construction of R_i , if $f(j) = i$ occurs, then put $(j + 2, 1), (1, j + 2) \in R_i$. Otherwise, put $(j + 2, j + 2) \in R_i$. Then $\langle B_n, G_n, R_n, 1 \rangle_{n \in \mathbb{N}}$ satisfies our assumptions, so has a sequence $\langle M_n \rangle_{n \in \mathbb{N}}$ of solutions by $\text{Seq}(B'_{\text{Hcb}}, G' \text{-M})$. It is easy to see that $V := \{i : (0, 0) \in M_i\}$ is the range of f . \square

Remark 3.4. It has been recently established in [15] and [5] that for a Π_2^1 statement of a certain syntactical form, its provability in (semi-)intuitionistic systems guarantees the provability of its sequential form in weak subsystems of second order arithmetic.³ Such kind of results are called “uniformization theorems”. The first uniformization theorems in higher order setting [15] can be applied for Π_2^1 statements of the following syntactical form:

$$(\spadesuit) \quad \forall X (\varphi(X) \rightarrow \exists Y \psi(X, Y)),$$

where $\varphi(X)$ is purely universal and $\psi(X, Y)$ is in sufficiently large class Γ_2 of formulas. On the other hand, Dorais has recently shown other uniformization theorems in second order setting [5]. The advantage of Dorais’s uniformization theorems compared to the former is that it can be applied for more Π_2^1 statements, namely, for Π_2^1 statements of the form (\spadesuit) with $\varphi(X)$ including purely existential formulas as subformula. (See [5, Section 4] for details.) By a careful inspection, one can check that the assertion “a bipartite graph $(B, G; R)$ satisfies the constant bounded Hall condition via k ” is (in intuitionistic sense) formalized as a formula of form $\forall x \exists y A_0(B, G, R, k)$ where A_0 is quantifier-free. That is to say, Dorais’s uniformization theorems can be applied to our marriage theorems with the constant bounded Hall condition while the uniformization theorems in [15] can not. As a consequence of Theorem 3.2 and 3.3, we have the following. (Note that EL is the intuitionistic second order system and RCA consists of EL and the law of excluded middle. See [5] for the definition of each symbol.)

- (1) $B'_{\text{Hcb}} \text{G-M}$ and $B'_{\text{Hcb}} \text{G}' \text{-M}$ are not provable in $\text{EL} + \text{WKL} + \text{GC}_L + \text{CN}_L$.
- (2) $B'_{\text{Hcb}} \text{G-M}$ and $B'_{\text{Hcb}} \text{G}' \text{-M}$ are not provable in $\text{EL} + \text{GC} + \text{CN}$.

3.2.2. Recursive Construction. By inspecting the proofs of Theorem 3.2 and Theorem 3.3, one notices that even if the Hall condition is bounded by $k = 1$, the marriage problem is not solvable uniformly in RCA. By contrast, if the marriage problem is G -highly recursive, then it is uniformly solvable in RCA_0 , regardless of the size of the constant bound on the Hall condition.

Theorem 3.5. $\text{RCA}_0 \vdash \text{Seq}(B'_{\text{Hcb}}, G'' \text{-M})$, that is, the following is provable in RCA_0 : For any sequence $\langle B_n, G_n, R_n, p_n, k_n \rangle_{n \in \mathbb{N}}$ such that $R_n(B_n, G_n)$ is G_n -highly recursive via p_n and satisfies the constant bounded Hall condition via k_n , there exists a sequence $\langle M_n \rangle_{n \in \mathbb{N}}$ of the solutions.

Proof. Rather than formally proving the sequential form, we give a uniform proof in RCA_0 of $B'_{\text{Hcb}} \text{G}'' \text{-M}$ for a graph with B infinite. This proof can easily be transformed into a proof of $\text{Seq}(B'_{\text{Hcb}}, G'' \text{-M})$ in RCA_0 .

Let $R(B, G)$ be a bipartite graph which is G -highly recursive and satisfies the constant bounded Hall condition via k , and $\{b_i : i \in \mathbb{N}\}$ be an enumeration of B . We shall now construct a solution of $R(B, G)$ by a recursive procedure. Let $\theta(u, v)$ say that u encodes a sequence $\langle u_i \rangle_{i < v+1}$ of length $v + 1$ of chains $u_i = \langle u_{i,j}^B, u_{i,j}^G \rangle_{j < \text{lh}(u_i)}$, where each u_i is a least non-proper R_i -chain (Definition 2.9) of finite length in the remaining graph $R_i(B_i, G_i) = R(B, G) - \bigcup_{i' < i} u_{i'}$

³Precisely, the systems employed in [15] and [5] are based not on set-based language but on function-based language. However, we do not emphasize it here, because the structure of our statements allows direct translations between set-based and function-based formalizations.

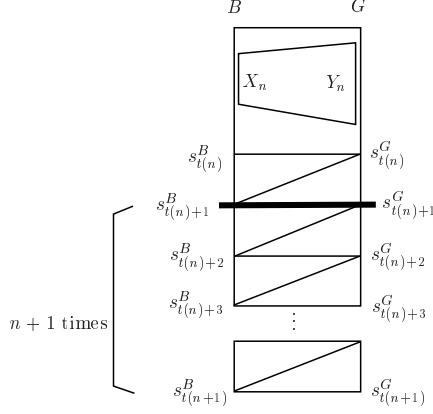


FIGURE 4. A proper R -chain s in the proof of Claim 3.6

and b_i is contained in $\bigcup_{i' \leq i} u_{i'}^B$. Note that the present u_i is not a chainable matching as in the proof of Theorem 2.7. Now $\theta(u, v)$ is written as a Σ_0^0 formula because $R(B, G)$ is G -highly recursive.

Suppose that we have shown $\forall v \exists u \theta(u, v)$. Then there exists a function which outputs the unique u^v for each $v \in \mathbb{N}$ by Δ_1^0 comprehension as in the proof of Theorem 2.7. Now we construct a function g by the following primitive recursion in RCA_0 .

$$\begin{aligned} g(0) &= \text{the least matching of } ((u^0)_0)^B \text{ in } R, \\ g(v+1) &= \text{the least matching of } ((u^{v+1})_{v+1})^B \text{ in } R - \bigcup_{i \leq v} u_i. \end{aligned}$$

This primitive recursion works by the finite marriage theorem, since the definition of $\theta(u, v)$ ensures the Hall condition for each subgraph $(u^{v+1})_{v+1}$ in each remaining graph. We take $M := \bigcup_{v \in \mathbb{N}} g(v)$, then we can straightforwardly verify in RCA_0 that M is an injection from B to G . Thus, it suffices to show $\forall v \exists u \theta(u, v)$ by Σ_1^0 induction on v . To show $\exists u \theta(u, 0)$, we first show the following key claim.

Claim 3.6 (RCA_0). If a bipartite graph $R(B, G)$ satisfies the constant bounded Hall condition via $k \in \mathbb{N}$, then there is no proper R -chain $s = \langle s_i^B, s_i^G \rangle_{i < lh(s)}$ of length more than $t(k+1)$, where $t(k) := k(k+3)/2$.

(*Proof of Claim.*) Suppose not, i.e., assume that $s = \langle s_i^B, s_i^G \rangle_{i < lh(s)}$ be a proper R -chain of length more than $t(k+1)$. Note that $t(k+1) - (t(k)+1) = k+1$. Now we shall show that for all $n \leq k+1$ there exists $X \subseteq s_{t(n)}^B$ and $Y \subseteq s_{t(n)}^G$ such that $Y \subseteq R[X]$ and $|X| + n \leq |Y|$ by induction on n . Note that the above statement can be written as a Σ_0^0 formula with the use of s , then this induction can be carried out in our system RCA_0 . The initial step is accomplished obviously. Let X_n and Y_n be witnesses of the case of n , i.e., $X_n \subseteq s_{t(n)}^B$, $Y_n \subseteq s_{t(n)}^G$, $Y_n \subseteq R[X_n]$ and $|X_n| + n \leq |Y_n|$ hold. By the properness of R -chain s (see Fig. 4), we can choose $g_j \in s_j^G \setminus s_{j-1}^G \neq \emptyset$ for each $t(n) < j \leq t(n+1)$.

- In the case that $s_{t(n)+1}^B \cap R^{-1}[g_{j_1}] = \emptyset$ for some $t(n)+1 < j_1 \leq t(n+1)$.
Hence, $R[s_{t(n)+1}^B] \cap \{g_{j_1}\} = \emptyset$. Then, $g_{j_1} \in s_{j_1}^G \subseteq R[s_{j_1}^B]$ implies that there is $\hat{b} \in s_{j_1}^B \setminus s_{t(n)+1}^B$ such that $g_{j_1} \in R(\hat{b})$. Now $\hat{b} \in s_{j_1}^B \setminus s_{t(n)+1}^B = R^{-1}[s_{j_1-1}^G] \setminus R^{-1}[s_{t(n)}^G]$ implies that there is $\hat{g} \in s_{j_1-1}^G \setminus s_{t(n)}^G$ such that $\hat{b} \in R^{-1}(\hat{g})$. As $g_{j_1} \notin s_{j_1-1}^G$, the girls g_{j_1} and \hat{g} are different, and they are not contained in $s_{t(n)}^G$. Hence, the boy $\hat{b} \notin X_n \subseteq s_{t(n)}^B$ knows two different girls $g_{j_1}, \hat{g} \notin Y_n \subseteq s_{t(n)}^G$. Therefore, for $X_{n+1} := X_n \cup \{\hat{b}\}$ and $Y_{n+1} := Y_n \cup \{g_{j_1}, \hat{g}\}$, we have $Y_{n+1} \subseteq R[X_{n+1}]$ and $|X_{n+1}| + n + 1 \leq |Y_{n+1}|$.
- Otherwise, i.e., $s_{t(n)+1}^B \cap R^{-1}[g_j] \neq \emptyset$ for every $t(n)+1 < j \leq t(n+1)$.

Choose $x_j \in s_{t(n)+1}^B \cap R^{-1}[g_j]$ for each $t(n)+1 < j \leq t(n+1)$, and put $\hat{X} = \{x_j\}_{t(n)+1 < j \leq t(n+1)}$. Since $(s_{t(n)+1}^B, s_{t(n)+1}^G)$ satisfies the Hall condition, there exists $\hat{Y} \subseteq s_{t(n)+1}^G$ such that $|\hat{X}| \leq |\hat{Y}|$ holds. Then one can verify that \hat{X} and $\hat{Y} \cup \{g_j\}_{t(n)+1 < j \leq t(n+1)}$ are witnesses of the case of $n+1$ straightforwardly.

Therefore the induction step is also accomplished. Then there exists $X \subseteq s_{t(k)}^B$ and $Y \subseteq s_{t(k)}^G$ such that $Y \subseteq R[X]$ and $|X| + k < |Y|$ holds. This contradicts our assumption that $R(B, G)$ satisfies the constant bounded Hall condition via k and completes the proof of our claim. \square

Because $R(B, G)$ is G -highly recursive, we can effectively produce a non-proper R -chain s with starting point $b_0 \in B$ by the following procedure: Let s_0^B be the set consisting only of b_0 , take the first witnessed set of girls s_j^G such that $\langle s_{j'}^B, s_{j'}^G \rangle_{j' \leq j}$ forms an R -chain, and put $s_{j+1}^B = R^{-1}[s_j^G]$. Claim 3.6 ensures that this procedure would stop eventually until j is up to $t(k+1)$, i.e., $\langle s_j^B, s_j^G \rangle_{j \leq t(k+1)}$ is non-proper. Then, by Σ_0^0 least number principle, there exists u_0 such that $\theta(u_0, 0)$ holds. Thus the initial step is accomplished.

Next we turn to the induction step. Assume that $\exists u \theta(u, v)$ holds, and let u' be u such that $\theta(u, v)$ holds. Then $R' = R - \bigcup_{j \leq v} u'_j$ satisfies the constant bounded Hall condition by the disjoint property (1). As in the initial step, we can effectively produce a non-proper R' -chain s' of finite length, where we take b_{v+1} as the starting point of s' if $b_{v+1} \notin (u'_v)^B$. Let u_{v+1} be such a least s' , then $\theta(u' \hat{\ } u_{v+1}, v+1)$ holds. This completes the proof of our theorem. \square

Corollary 3.7. $\text{Seq}(B''_{\text{Hcb}}, G''\text{-M})$ is provable in RCA_0 .

4. WEIHRAUCH DEGREES

4.1. Basic Terminology. Many theorems of mathematics can be formalized as Π_2^1 sentences. In particular, marriage theorems can be written as the following Π_2^1 sentences:

$$(\forall R) [\varphi(R) \rightarrow \exists M \psi(R, M)],$$

where $\varphi(R)$ denotes that R is a graph with $B^{(\cdot)}, G^{(\cdot)}, H^{(\cdot)}$, and $\psi(R, M)$ denotes that M is a matching of R . Importantly, such a Π_2^1 theorem can be viewed as a (partial) multi-valued function $f : \subseteq \mathcal{P}(\mathbb{N}) \rightrightarrows \mathcal{P}(\mathbb{N})$ with an arithmetical domain, by interpreting $\psi(R, M)$ as $M \in f(R)$. Every single-valued selection s of f can be thought of as a witness of the Π_2^1 theorem, that is, s witnesses $\psi(R, s(R))$. The following reducibility notion is useful to estimate how difficult it is to find a witness of a given Π_2^1 theorem.

Definition 4.1 (See also [2, 3].). For multi-valued functions f and g , f is *Weihrauch reducible* to g (denoted $f \leq_W g$) if there are computable functions H, K such that $K(\text{id}, GH)$ is a single-valued selection of f for every single-valued selection G of g .

Let us consider the following partial multi-valued functions.

$$\begin{aligned} B_{\text{H}^{(\cdot)}}^{(\cdot)} G^{(\cdot)}\text{-M}(R) &= \{M : M \text{ is a matching of } R\}, \\ \text{dom}(B_{\text{H}^{(\cdot)}}^{(\cdot)} G^{(\cdot)}\text{-M}) &= \{R : R \text{ is a graph with } B^{(\cdot)}, G^{(\cdot)}, H^{(\cdot)}\}, \\ \text{KL}(T) = \text{WKL}(T) &= \{P : P \text{ is an infinite path through } T\}, \\ \text{dom}(\text{KL}) &= \{T \subseteq \mathbb{N}^* : T \text{ is an infinite finitely branching tree}\}, \\ \text{dom}(\text{WKL}) &= \{T \subseteq \mathbb{N}^* : T \text{ is an infinite binary tree}\}, \\ \text{Lim}_X(\langle p_n \rangle_{n \in \mathbb{N}}) &= \lim_{n \rightarrow \infty} p_n, \text{ where } X \text{ is a topological space,} \\ \text{dom}(\text{Lim}_X) &= \{\langle p_n \rangle_{n \in \mathbb{N}} \in X^{\mathbb{N}} : \lim_n p_n \text{ converges}\}. \end{aligned}$$

Intuitively, $f \leq_W g$ means that, to find a solution to the problem $f(x)$, it suffices to find a solution y to $g(H(x))$, since $K(x, y)$ is a solution to $f(x)$. The Weihrauch degree of the identity function id is analogous to the Δ_1^0 comprehension axiom RCA in second order arithmetic. The limit function $\text{Lim}_{\mathbb{N}^{\mathbb{N}}}$ is analogous to the Σ_1^0 comprehension, that is equivalent to the arithmetical comprehension ACA in reverse mathematics. Every function $f \leq_W \text{Lim}_{\mathbb{N}^{\mathbb{N}}}$ is said to be *computable with finitely many mind changes*, and $f \leq_W \text{Lim}_{\mathbb{N}^{\mathbb{N}}}$ is also said to be *limit computable* [1]. The function $\text{Lim}_{\mathbb{N}}$ is also known as the *discrete limit* [1]. The *parallelization* of a partial multi-valued function f is defined by $\widehat{f}(\langle x_i \rangle_{i \in \mathbb{N}}) = \prod_{i \in \mathbb{N}} f(x_i)$. Given a multi-valued function f_τ with an associated Π_2^1 theorem τ , its parallelization \widehat{f}_τ can be seen as the sequential version $\text{Seq}(\tau)$ of the original theorem τ . As shown in [2], WKL is Weihrauch equivalent to $\widehat{\text{LLPO}}$. As a counterpart of this result in the context of *constructive reverse mathematics* over intuitionistic analysis EL or intuitionistic finite type arithmetic HA^ω , Ishihara [16] has shown that WKL is equivalent to LLPO plus the axiom of countable choice for Π_1^0 disjunctions, $\Pi_1^0\text{-AC}^\vee$. It is also known that $\text{Lim}_{\mathbb{N}^{\mathbb{N}}} \equiv_W \widehat{\text{LPO}} \equiv_W \widehat{\text{Lim}_{\mathbb{N}}}$ holds (see [2]).

4.2. Weihrauch Degrees inside ACA. Contrary to the inequality $\text{Lim}_{\mathbb{N}^{\mathbb{N}}} <_W \text{KL}$ (obtained from the fact that KL is equivalent to weak König's lemma relative to the jump), the standard reverse mathematics [19] does not distinguish $\text{Lim}_{\mathbb{N}^{\mathbb{N}}}$ from KL since the collection of functions in every model of RCA_0 is closed under composition. Therefore, even if some theorem is shown to be equivalent to ACA over RCA_0 , the exact computational strength of the theorem has many possibilities including the jump, the double jump, and so on. The concept of Weihrauch degrees may help us to better understand the computational strength of Π_2^1 theorems.

Theorem 4.2. (1) All of the following multi-valued functions are Weihrauch equivalent to KL .

$$B'_H G\text{-M} \quad B'_H G'\text{-M} \quad B'_H G''\text{-M}$$

(2) All of the following multi-valued functions are Weihrauch equivalent to $\text{Lim}_{\mathbb{N}^{\mathbb{N}}}$.

$$\begin{array}{cc} B'_{H'} G\text{-M} & B'_{H''} G\text{-M} \\ B'_{H'} G'\text{-M} & B'_{H''} G'\text{-M} \\ B'_{H'} G''\text{-M} & \end{array}$$

(3) All of the following multi-valued functions are Weihrauch equivalent to WKL .

$$\begin{array}{ccc} B''_H G\text{-M} & B''_{H'} G\text{-M} & B''_{H''} G\text{-M} \\ B''_H G'\text{-M} & B''_{H'} G'\text{-M} & B''_{H''} G'\text{-M} \\ B''_H G''\text{-M} & B''_{H'} G''\text{-M} & \end{array}$$

(4) All of the following multi-valued functions are Weihrauch equivalent to id .

$$B_{H''} G''\text{-M} \quad B'_{H''} G''\text{-M} \quad B''_{H''} G''\text{-M}$$

Proof. (1) It suffices to show that $B'_H G\text{-M} \leq_W \text{KL} \leq_W B'_H G''\text{-M}$. It is easy to see that $B'_H G\text{-M} \leq_W \text{KL}$ since the set of all injective selections for a locally finite multi-valued function F forms a bounded $\Pi_1^{0,F}$ class. Conversely, given finitely branching tree T , we can effectively construct an instance of $B'_H G''\text{-M}$ whose solutions are computably homeomorphic to $\text{KL}(T)$ as follows. Put $B = T$, $G = T \setminus \{\langle \rangle\}$, and for each $\sigma \in T \setminus \{\langle \rangle\}$ enumerate $(\sigma, \sigma) \in R$ and $(\sigma^-, \sigma) \in R$, where σ^- is the unique immediate predecessor of σ .

(3) Straightforward. (4) By uniformity of the proof of the Theorem 2.7.

(2) By using $\text{Lim}_{\mathbb{N}^{\mathbb{N}}}$, $B'_{H'} G\text{-M}$ is reducible to $B''_{H''} G''\text{-M}$. Thus, by (4), we have $B'_{H'} G\text{-M} \leq_W \text{Lim}_{\mathbb{N}^{\mathbb{N}}}$. Conversely, by uniformity of proofs of Lemma 2.3 and 2.6, $\text{Lim}_{\mathbb{N}^{\mathbb{N}}}$ is Weihrauch reducible to $B'_{H'} G''\text{-M}$ and $B'_{H''} G'\text{-M}$. \square

4.3. Weihrauch Degrees inside RCA. If a Π_1^1 theorem τ is provable in RCA, then the associated multi-valued function f_τ is always expected to be non-uniformly computable. Here, a multi-valued function $f : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is said to be *non-uniformly computable* if there is a single-valued selection F of f such that $F(x)$ is computable in x for all $x \in \text{dom}(F)$. For instance, $\text{Lim}_{\mathbb{N}}$, LPO , LLPO , and id are non-uniformly computable. Over RCA, if its sequential version $\text{Seq}(\tau)$ is equivalent to an axiom, say WKL , then one may also guess that its parallelization \widehat{f}_τ is expected to be Weihrauch-equivalent to WKL . For instance, LLPO is such a multi-valued function, that is, LLPO is non-uniformly computable, and its parallelization $\widehat{\text{LLPO}}$ is Weihrauch equivalent to WKL . Unfortunately, however, LLPO is not a unique such function.

For a tree T , a string $\sigma \in T$ is *branching* if at least two immediate successors of σ are contained in T . Now we introduce the following multivalued functions.

$$\begin{aligned} \text{WKL}_{?<\omega}(T) &= \text{WKL}_{?n}(T) = \text{WKL}(T), \\ \text{dom}(\text{WKL}_{?n}) &= \{T \in \text{dom}(\text{WKL}) : T \text{ has less than } n \text{ branching nodes}\}, \\ \text{dom}(\text{WKL}_{?<\omega}) &= \{T \in \text{dom}(\text{WKL}) : T \text{ has only finitely many branching nodes}\}, \end{aligned}$$

Obviously $\text{LLPO} \leq_W \text{WKL}_{?2}$, and every $\text{WKL}_{?n}$ is Weihrauch reducible to an iterative use of sufficiently many LLPO 's. Hence, one can think of $\text{WKL}_{?<\omega}$ as a natural enrichment of LLPO . We now see the Weihrauch degree of constant bounded marriage theorems.

Theorem 4.3. (1) All of the following multi-valued functions are Weihrauch equivalent to $\text{Lim}_{\mathbb{N}}$.

$$B'_{H_{\text{cb}}} G\text{-M} \quad B'_{H_{\text{cb}}} G'\text{-M}$$

(2) All of the following multi-valued functions are Weihrauch equivalent to id .

$$B'_{H_{\text{cb}}} G''\text{-M} \quad B''_{H_{\text{cb}}} G''\text{-M}$$

Proof. (1) To see $B'_{H_{\text{cb}}} G\text{-M} \leq_W \text{Lim}_{\mathbb{N}}$, it suffices to check that the canonical indices of parameters c_1 , X_1 , X_2 , Y_1 , $R[X_1]$, and $R^{-1}[Y_1]$ in Theorem 3.1 are effectively determined by finitely many mind changes. At stage $s+1$, if we find a finite set X of boys such that $|R_s[X]| > |X| + c_{1,s}$, where $R_s[X] = R[X] \cap \{0, \dots, s\}$, then renew X_1 and c_1 , that is, put $X_{1,s+1} = X$ and $c_{1,s+1} = |R_s[X]| - |X|$. Clearly, $c_1 = \lim_s c_{1,s}$ and $X_1 = \lim_s X_{1,s}$ converge. If $c_{1,s}$ and $X_{1,s}$ are fixed, since X_2 and Y_1 have at most $c_{1,s}$ elements and $R^{-1}[Y_1]$ has at most $2c_{1,s}$ elements, we can determine the indices of X_2 and Y_1 by $4c_{1,s}$ many mind changes. Thus, if c is an upper bound for c_1 , then we can find a solution by $\sum_{d \leq c} 4d$ many mind changes.

Conversely, to see $\text{Lim}_{\mathbb{N}} \leq_W B'_{H_{\text{cb}}} G'\text{-M}$, given a sequence $p = (p_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, we construct a graph R_p with the constant bounded Hall condition. Put $B = G = \mathbb{N}$, and $(0, 0) \in R$ with a parameter $g_0 = 0$. If $p_{n+1} = p_n$, then put $(n+1, n+1) \in R$ and $g_{n+1} = g_n$. If $p_{n+1} \neq p_n$, then put $(n+1, g_n) \in R$, $(0, n+1) \in R$, and $g_{n+1} = n+1$. Then, it is easy to see that R has the unique matching M , and $p_{M(0)} = \lim_n p_n$.

(2) By the uniformity of the proof of Theorem 3.5. \square

Theorem 4.4. $\text{WKL}_{?<\omega} \leq_W \text{B''}_{\text{Hcb}} G'-\text{M} \leq_W \text{B''}_{\text{Hcb}} G-\text{M} \leq_W \text{inf}(\text{Lim}_{\mathbb{N}}, \text{WKL})$.

Proof. The inequality $\text{B''}_{\text{Hcb}} G-\text{M} \leq_W \text{inf}(\text{Lim}_{\mathbb{N}}, \text{WKL})$ clearly holds by Theorem 4.3 (1). Given a tree T and a nonempty string $\sigma \in T$, we denote by $l_T(\sigma)$ the longest initial segment of σ whose immediate predecessor is branching or empty. To see $\text{WKL}_{?<\omega} \leq_W \text{B''}_{\text{Hcb}} G'-\text{M}$, assume that T is an infinite tree with only finitely many branching nodes. Put $B = T$ and $G = T \setminus \{\langle \rangle\}$. For each $\sigma \in T$, if σ is not a leaf, then enumerate $(\sigma, \tau) \in R$ for each immediate successor $\tau \in T$ of σ . If a nonempty string $\sigma \in T$ is a leaf or branching, then we also enumerate $(\sigma, l_T(\sigma)) \in R$. Note that every non-branching boy knows just one girl, and every nonempty branching boy knows just three girls. Hence, if T has only finitely many branching nodes, then $R(B, G)$ is constant bounded. If τ is of the form $l_T(\sigma)$ such that σ is a leaf or branching, then $\tau = l_T(\sigma)$ is known by at most two boys σ and $l_T(\sigma)^-$. Here note that $l_T(\sigma)^-$ must be branching. Otherwise, τ is known by at most one boy. Hence, $R(B, G)$ satisfies the Hall condition. Given any matching of $R(B, G)$, we can effectively find an infinite path through T . \square

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(M. Fujiwara) MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, 6-3, ARAMAKI AOKA, AOKA-KU, SENDAI, MIYAGI, JAPAN.
E-mail address: sb0m29@math.tohoku.ac.jp

(K. Higuchi) DEPARTMENT OF MATHEMATICS AND INFORMATICS, FACULTY OF SCIENCE, CHIBA UNIVERSITY, 1-33 YAYOI-CHO, INAGE, CHIBA, JAPAN.
E-mail address: khiguchi@g.math.s.chiba-u.ac.jp

(T. Kihara) JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY, 1-1 ASAHIDAI, NOMI, ISHIKAWA, JAPAN.
E-mail address: kihara@jaist.ac.jp