A COMPARISON OF VARIOUS ANALYTIC CHOICE PRINCIPLES

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ABSTRACT. We investigate computability theoretic and descriptive set theoretic contents of various kinds of analytic choice principles by performing a detailed analysis of the Medvedev lattice of Σ^1_1-closed sets. Among others, we solve an open problem on the Weihrauch degree of the parallelization of the Σ^1_1-choice principle on the integers. Harrington's unpublished result on a jump hierarchy along a pseudo-well-ordering plays a key role in solving this problem.

1. INTRODUCTION

1.1. **Summary.** The study of the Weihrauch lattice aims to measure the computability theoretic difficulty of finding a choice function witnessing the truth of a given $\forall\exists$-theorem (cf. [3]) as an analogue of reverse mathematics [19]. In this article, we investigate the uniform computational contents of the axiom of choice Σ^1_1-AC and dependent choice Σ^1_1-DC for Σ^1_1 formulas in the context of the Weihrauch lattice.

The computability-theoretic strength of these choice principles is completely independent of their proof-theoretic strength, since the meaning of an impredicative notion such as Σ^1_1 is quite unstable among models of second-order arithmetic. Nevertheless, it is still interesting to examine the uniform computational contents of Σ^1_1-AC and Σ^1_1-DC in the full model $\mathcal{P}\mathbb{N}$. For instance, this setting is particularly relevant for descriptive set theory and related areas, and indeed, the complexity of the axiom of choice has already been studied a lot in descriptive set theory, under the name of uniformization.

For a set $A \subseteq X \times Y$ define the *x-th section of* A as $A(x) = \{y \in Y : (x, y) \in A\}$. We say that a partial function $g\colon\subseteq X \to Y$ is a *choice function* for A if $g(x)$ is defined and $g(x) \in A(x)$ whenever $A(x)$ is nonempty. Such a choice function is also called a *uniformization* of A. In descriptive set theory and related areas, there are a number of important results on measuring the complexity of choice functions: Let X and Y be standard Borel spaces. The Jankov-von Neumann uniformization theorem (cf. [13, Theorem 18.1]) states that if A is analytic, then there is a choice function for A which is measurable w.r.t. the σ-algebra generated by the analytic sets. The Luzin-Novikov uniformization theorem (cf. [13, Theorem 18.10]) states that if A is Borel each of whose section is at most countable, then there is a Borel-measurable choice function for A. Later, Arsenin and Kunugui (cf. [13, Theorem 35.46]) showed that the same holds even if each section is allowed to be σ-compact.

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In our setting, (a Σ^1_1 definition φ_A of) a set $A \subseteq X \times Y$ is considered to be an instance of the "uniformization problem", and any choice function $g_A :\subseteq X \to Y$ for A to be a solution of the instance. Then we examine the degree of difficulty of such a transformation $\varphi_A \mapsto g_A$. In other words, we investigate not only the complexity of a choice function g_A itself, but also of the uniform content $\varphi_A \mapsto g_A$ of the uniformization problem. By the technique of relativization one can solve problems in this area with results about uniform choice. Also, uniform results can be more precise and one can wonder whether results about non-uniform choice are still true in their uniform version.

The main tool for comparing these degrees of difficulty is Weihrauch reduction. The notion of Weihrauch degree is used as a tool to classify certain $\forall\exists$-statements by identifying $\forall\exists$-statements with a partial multivalued function. Informally speaking, a (possibly false) statement $S \equiv \forall x \in X\ [Q(x) \to \exists y P(x,y)]$ is transformed into a partial multivalued function $f :\subseteq X \rightrightarrows Y$ such that $\mathrm{dom}(f) = \{x : Q(x)\}$ and $f(x) = \{y : P(x,y)\}$. Then, measuring the degree of difficulty of witnessing the truth of S is identified with that of finding a choice function for f. Here, we consider choice problems for partial multivalued functions rather than relations in order to distinguish the hardest instance $f(x) = \emptyset$ and the easiest instance $x \in X \setminus \mathrm{dom}(f)$.

In this article, we only consider subspaces of $\mathbb{N}^{\mathbb{N}}$, so we can use the following version of Weihrauch reducibility. For partial multivalued functions f, g, we say that f is *Weihrauch reducible to* g (written $f \leq_{\mathsf{W}} g$) if there are partial computable functions h, k such that $x \mapsto k(x, G \circ h(x))$ is a choice for f whenever G is a choice for g. In other words,

$$(\forall x \in \mathrm{dom}(f))(\forall y)\ [y \in g(h(x)) \implies k(x,y) \in f(x)].$$

In recent years, a lot of researchers has employed this notion to measure uniform computational strength of $\forall\exists$-theorems in analysis as an analogue of reverse mathematics. Roughly speaking, the study of the Weihrauch lattice can be thought of as "reverse mathematics plus uniformity minus proof theory." But this disregard for proof theory provides us a new insight into the classification of impredicative principles as we see in this article. For more details on the Weihrauch lattice, we refer the reader to a recent survey article [3].

Coming back to our study of analytic axioms of choice, we write $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to X}$ for the independent axiom of countable choice on X seen as a partial multivalued function. As the countable axiom of choice corresponds to countably many independent choice, it is noted in Observation 2.1 that this Weihrauch problem corresponds to the parallelization of the Σ^1_1 single choice, $\widehat{\Sigma^1_1\text{-}\mathsf{C}_X}$. In particular, $\Sigma^1_1\text{-}\mathsf{AC}_2 \equiv_{\mathsf{W}} \widehat{\Sigma^1_1\mathsf{C}_2}$. The dependent choice $\Sigma^1_1\text{-}\mathsf{DC}_X$ corresponds to finding a path through a Σ^1_1 tree, where a finite path contains the choices already made, and the possible extensions the choices to come. For instance, we note in Observation 2.3 that $\Sigma^1_1\text{-}\mathsf{DC}_2 \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{WKL}$, the problem of finding an infinite path in a binary Σ^1_1 tree (see Section 2.1). We have the following, proved in Proposition 2.5:

Fact 1.1. We have $\Sigma^1_1\text{-}\mathsf{DC}_2 \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to 2}$.

However, using the equivalences noted in the above paragraph, Lemma 4.7 in [14] asserts:

Fact 1.2 (Kihara-Marcone-Pauly [14, Lemma 4.7])**.** In contrast to the previous fact, we have $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}} \not\leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_2$.

This suggests the following question about dependent and independent choice when the objects are chosen from subsets of the integers:

Question 1.3 (Brattka et al. [2] and Kihara et al. [14, Question 4.10])**.** Do we have $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$?

Note that this question was not asked in these terms in [2] and [14, Question 4.10], however the two formulation are equivalent by Proposition 2.2.

To negatively solve this question, we will employ the notion of a pseudo-hierarchy: A remarkable discovery by Harrison is that some *non*-well-ordering $\prec$ admits a transfinite hierarchy based on an arithmetical formula. Furthermore, a basic observation is that, without deciding if a given countable linear ordering $\prec$ is well-ordered or not, one can either produce an arithmetical transfinite hierarchy along $\prec$ or construct an infinite $\prec$-decreasing sequence. Indeed, we will see that the degree of difficulty of such a construction is quite close to that of uniformizing analytic sets with compact sections, which is drastically easier than deciding well-orderedness of a countable linear ordering.

In conclusion, we have an interesting difference between countable choice on 2 and on $\mathbb{N}$: In the former, independent and dependent choices correspond, while in the latter they differs. One can wonder when this transition happens, for various restrictions of the set we choose from. Many restrictions on the principle of choice on a single set have already been studied, as for instance those defined in [3, Definition 7.4]. In this article, we will study the axioms of dependent and independent countable choices for the restrictions to *finite*, *cofinite*, *all-or-finite*, *all-or-unique* and *finite-or-cofinite* sets of natural numbers. In summary, we will show the following in the Weihrauch context:

- Countable choice on finite Σ^1_1 sets is strictly easier than countable choice on all-or-finite Σ^1_1 sets (Corollary 3.13).
- Countable choice on cofinite Σ^1_1 sets is incomparable with countable choice on (all-or-)finite Σ^1_1 sets (Corollary 3.19).
- Countable choice on all-or-finite Σ^1_1 sets is strictly easier than countable choice on Σ^1_1 sets (Corollary 3.19).
- Countable choice on finite Σ^1_1 sets has the same difficulty (modulo arithmetical equivalence) as some disjunctive form of arithmetical transfinite recursion (Theorem 2.12).
- Countable choice on (all-or-)finite Σ^1_1 sets has the same difficulty as dependent choice on (all-or-)finite Σ^1_1 sets (Theorem 3.3 and Theorem 3.10).
- Countable choice on Σ^1_1 sets is strictly easier than dependent choice on Σ^1_1 sets (Theorem 3.30).

1.2. **Preliminaries.**

Weihrauch reducibility. We use several operations on the Weihrauch lattice (see also [3, 4]). Given a partial multivalued function f, the *parallelization of* f is defined as follows:

$$\widehat{f}((x_n)_{n\in\mathbb{N}}) = \prod_{n\in\mathbb{N}} f(x_n) = \{(y_n)_{n\in\mathbb{N}} : (\forall n)\ y_n \in f(x_n)\}.$$

If $f \equiv_{\mathsf{W}} \hat{f}$, then we say that f is *parallelizable*. Given partial multivalued functions f and g, the *compositional product of* f *and* g (written $g \star f$) is a function which realizes the greatest Weihrauch degree among $g_0 \circ f_0$ for $f_0 \leq_{\mathsf{W}} f$ and $g_0 \leq_{\mathsf{W}} g$.

It is known that such an operation $\star$ exists. For basic properties of parallelization and compositional product, see also [4].

Analytic sets as co-enumeration. In this article, we are mainly interested in analytic sets, in other words in sets that are Σ^1_1 relative to some oracle A. It is well-known that $\Sigma^1_1(A)$ sets can be seen as a co-enumeration process along ω_1^A (the supremum of all A-computable ordinals), where at each step the co-enumeration is $\Delta^1_1(A)$ in a uniform manner. This is one of the most fundamental ideas in the study of Σ^1_1 sets; see any textbook on hyperarithmetic theory or higher computability theory, cf. [12].

This fact essentially follows from the Spector-Gandy theorem, i.e., $\Pi_1(L_{\omega_1^{\mathrm{CK}}}) = \Sigma^1_1$. More precisely, $P \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^1_1 if and only if there is a Π_1 formula φ in the language of set theory such that $X \in P \iff L_{\omega_1^X}[X] \models \varphi(X)$. Obviously, this idea is *uniformly relativizable* to any oracle by applying the above equivalence to a universal Σ^1_1 set which parametrizes all analytic sets (and then consider its cross-sections).

One can utilize this fact to describe a construction of a $\Sigma^1_1(A)$ set as a uniform (co-enumeration) algorithm along an ω_1^A-step computation. There are a lot of recursion-theoretic frameworks to rigorously describe the idea of ordinal step computations, such as admissible sets, norms, and inductive operators; see e.g. [12, 17]. In particular, we use the following notions:

A Π^1_1-*norm* on a Π^1_1 set $P \subseteq X$, where X is either $\mathbb{N}$ or $\mathbb{N}^{\mathbb{N}}$, is a map $\varphi\colon X \to \omega_1 \cup \{\infty\}$ such that $\forall x \in X, (\varphi(x) < \omega_1^x \vee \varphi(x) = \infty)$, such that $P = \{x \in X : \varphi(x) < \infty\}$ and such that the following relations $\leq_\varphi$ and $<_\varphi$ are Π^1_1:

$$\begin{aligned} a \leq_\varphi b &\iff \varphi(a) < \infty \text{ and } \varphi(a) \leq \varphi(b), \\ a <_\varphi b &\iff \varphi(a) < \infty \text{ and } \varphi(a) < \varphi(b). \end{aligned}$$

In our case, we will focus on the case where $X = \mathbb{N}$. In this case, φ has value in $\omega_1^{\mathrm{CK}} \cup \{\infty\}$. It is well-known that every Π^1_1 set admits a Π^1_1-norm (in an effective, uniform, manner): Consider a many-one reduction from a Π^1_1 set P to the set WO of well orderings. Then we define $P_\alpha = \{n : \varphi(n) < \alpha\}$, which is a Δ^1_1 set, and if α is limit then we have $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$. If $n \in P_\alpha$, it is natural to say that n *is enumerated into P by step α*. Similarly, a Σ^1_1 set S can be written as the intersection $S = \{S_\alpha : \alpha < \omega_1^{\mathrm{CK}}\}$ of a decreasing sequence of Δ^1_1 sets, and if $n \notin S_\alpha$, we say that n *is removed from S by step α*. We often use some other sentences which have similar meanings.

This argument is uniformly relativizable to any oracle by using a Π^1_1 norm on a Π^1_1 set which parametrizes all coanalytic sets (cf. [17]). In particular, every $\Sigma^1_1(A)$ set can uniformly be seen as a co-enumeration of $\Delta^1_1(A)$ sets, of length ω_1^{CK}. In the case of a Σ^1_1 sets $P \subseteq \mathbb{N}^{\mathbb{N}}$, P can be seen as a coenumeration of length ω_1, where an element x can only be removed before stage ω_1^x.

The reversal is also well-known (which is essentially Kleene's HYP quantification, whose uniform version is also clearly effective). This fact can also be presented as an ordinal step construction as follows: If we describe a Δ^1_1 rule $\Gamma\colon P_{<\alpha} \mapsto P_\alpha$, which satisfies the property $R \subseteq \Gamma(R)$, then the least fixed point P of Γ is Π^1_1; and indeed the iteration $(\Gamma^\alpha(\emptyset))_\alpha$ always stabilizes to the fixed point at ω_1^{CK} steps, i.e., $P = \bigcup\{P_\alpha : \alpha < \omega_1^{\mathrm{CK}}\}$. Such a Γ is known as a Δ^1_1 *inductive operator*, which is a fundamental notion in generalized recursion theory, cf. [12]; see also [17, Section

7C] for uniformity. A typical example of a Δ^1_1 inductive operator is the standard construction of the H-hierarchy of Turing jumps (see also Section 2.4). Similarly, if we describe a Δ^1_1 rule $\Theta\colon S_{<\alpha} \mapsto S_\alpha$ with $\Theta(R) \subseteq R$ we eventually obtain a Σ^1_1 set $S = \bigcap\{S_\alpha : \alpha < \omega_1^{\mathrm{CK}}\}$. We consider $S_\alpha = \Theta^\alpha(\mathbb{N}^{<\mathbb{N}})$ as the *stage α approximation of S* in our Δ^1_1-construction of a Σ^1_1 set S. Again, this is *uniformly relativizable* as above.

There is an important result of recursion theory, known as Spector's Σ^1_1 bounding principle (admissibility of ω_1^{CK}). This theorem says that during an enumeration, on certain conditions, an event will already happen at some stage of computation (and not only at stage ω_1^A, outside the scope of the enumeration).

Theorem 1.4 (Spector's bounding principle)**.** *If $f\colon \mathbb{N} \to \mathcal{O}^A$ is a $\Sigma^1_1(A)$ definable function from $\mathbb{N}$ to the ordinals, then it must be bounded strictly below ω_1^A. Here, $\mathcal{O}^A$ defines the set of codes for ordinals recursive in A.*

We will use this way of defining Σ^1_1 sets together with Spector's bounding principle quite often in this paper. For instance, let us show that there exists a computable function f such that $f(a)$ is an index for a Σ^1_1 set $S \subseteq \mathbb{N}$ such that:

- if a is an index for a nonempty Σ^1_1 set, then $S = \mathbb{N}$,
- if a is an index for an empty Σ^1_1 set, then $S = \emptyset$.

Indeed, let a be an index for a Σ^1_1 set $E \subseteq \mathbb{N}$, and define $f(a)$ to be the index of the Σ^1_1 set defined by the following co-enumeration: at stage α, do nothing if E_α (E at stage α) is not empty, and remove everything if $E_\alpha = \emptyset$.

Let S_a be the set of index $f(a)$. If S_a is empty, then at some stage α, E_α is empty, so $E = \emptyset$. Otherwise, suppose that E is empty. Then, we claim that there must exists a stage α at which E_α is already empty: indeed, consider the function which to n associates the stage where n is removed is Σ^1_1, and by Spector's bounding principle (Theorem 1.4) let $\alpha < \omega_1^{\mathrm{CK}}$ be a bound to it. Then, at stage α, E_α is empty and therefore S_a is also empty. So S_a is empty if and only if E_α is empty. To conclude, it is clear from the definition of S_a that if it is nonempty, then $S_a = \mathbb{N}$.

An easy but important observation is that this construction is uniformly relativizable. To proceed the construction (of an inductive operator), we only need the existence of a bound $\alpha < \omega_1^A$, but do not need to know the value of such an α, and this fact is ensured by Theorem 1.4; that is, we do not require a uniform version of Spector's bounding principle, although it is not hard to see that Spector's bounding principle is uniformly relativizable.

2. Equivalence results in the Weihrauch lattice

2.1. **Σ^1_1-Choice Principles.** One of the main notions in this article is the Σ^1_1-choice principle. In the context of the Weihrauch degrees, the Σ^1_1-choice principle on a space X is formulated as the partial multivalued function which, given a code of a nonempty analytic set A, chooses an element of A.

We fix a coding system of all analytic sets in a Polish space X, and let S_p be the analytic subset of X coded by $p \in \mathbb{N}^\mathbb{N}$. For instance, let S_p be the projection of the p-th closed subset of $X \times \mathbb{N}^\mathbb{N}$ (i.e., the complement of the union of all basic open balls of index $p(n)$ for some n) into the first coordinate (cf. [14]). Such a p is called an *analytic code* (or a *Σ^1_1-name*) of S_p and a *coanalytic code* (or a *Π^1_1-name*) of the complement of S_p.

The Σ^1_1-*choice principle on* X, Σ^1_1-C_X, is the partial multivalued function which, given a code of a nonempty analytic subset of X, chooses one element from X. Formally speaking, it is defined as the following partial multivalued function:

$$\mathrm{dom}(\Sigma^1_1\text{-}\mathsf{C}_X) = \{p \in \mathbb{N}^\mathbb{N} : S_p \neq \emptyset\},$$
$$\Sigma^1_1\text{-}\mathsf{C}_X(p) = S_p.$$

For the basics on the Σ^1_1-choice principle on X, see also [14]. In a similar manner, one can also consider the Γ-choice principle on X, Γ-C_X, for any represented space X and any collection Γ of subsets of X endowed with a representation $S :\subseteq \mathbb{N}^\mathbb{N} \to \Gamma$ (where we write S_p in place of $S(p)$). We first describe how this choice principle is related to several very weak variants of the axiom of choice.

In logic, the *axiom of* Σ *choice*, Σ-AC, is known to be the following statement:

$$\forall a \exists b\ \varphi(a,b) \ \longrightarrow\ \exists f \forall a\ \varphi(a, f(a)),$$

where φ is a Σ formula. If we require $a \in X$ and $b \in Y$, the above statement is written as Σ-$\mathsf{AC}_{X\to Y}$. We examine the complexity of a procedure that, given a Σ^1_1 formula φ (with a parameter) satisfying the premise of Σ^1_1-$\mathsf{AC}_{X\to Y}$, returns a choice for φ. In other words, we interpret Σ^1_1-$\mathsf{AC}_{X\to Y}$ as the following partial multivalued function:

$$\mathrm{dom}(\Sigma^1_1\text{-}\mathsf{AC}_{X\to Y}) = \{p \in \mathbb{N}^\mathbb{N} : \forall a \exists b\ \langle a, b\rangle \in S_p\},$$
$$\Sigma^1_1\text{-}\mathsf{AC}_{X\to Y}(p) = \{f \in Y^X : (\forall a)\ \langle a, f(a)\rangle \in S_p\}.$$

Unfortunately, this interpretation is different from the usual (relative) realizability interpretation. However, the above interpretation of Σ^1_1-$\mathsf{AC}_{X\to\mathbb{N}}$ is related to a descriptive-set-theoretic notion known as the generalized reduction property (or equivalently, the number uniformization property) for Σ^1_1 (cf. [13, Definition 22.14]).

The equivalence Σ^1_1-$\mathsf{C}_X \equiv_\mathsf{W} \Sigma^1_1$-$\mathsf{AC}_{1\to X}$ is obvious. In this article, we are mainly interested in *countable choice* Σ-$\mathsf{AC}_{\mathbb{N}\to X}$. The countable choice principles Σ-$\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ and Σ-$\mathsf{AC}_{\mathbb{N}\to\mathbb{N}^\mathbb{N}}$ are also known as Σ-$\mathsf{AC}_{0,0}$ and Σ-$\mathsf{AC}_{0,1}$, respectively. In the context of Weihrauch degrees, the interpretation of the countable choice, Σ^1_1-$\mathsf{AC}_{\mathbb{N}\to X}$, is obviously related to the parallelization of the Σ^1_1-choice principle.

Observation 2.1. If X is an initial segment of $\mathbb{N}$, then we have $\widehat{\Sigma^1_1\text{-}\mathsf{C}_X} \equiv_\mathsf{W} \Sigma^1_1$-$\mathsf{AC}_{\mathbb{N}\to X}$. □

In logic, the *axiom of* Σ^1_1-*dependent choice on* X is the following statement:

$$\forall a \exists b\ \varphi(a,b) \ \longrightarrow\ \forall a \exists f\ [f(0) = a \ \&\ \forall n\ \varphi(f(n), f(n+1))],$$

where φ is a Σ^1_1-formula, and a and b range over X. It is known that dependent choice is equivalent to the statement saying that if T is a definable pruned tree of height ω, then there is an infinite path through T. However, this is achieved by using dependent choice on the set of finite strings of elements of X, $X^{<\mathbb{N}}$. As this is the principle that is actually used, and the one that makes sense for X being finite, this is how we define Σ^1_1-DC_X:

$$\mathrm{dom}(\Sigma^1_1\text{-}\mathsf{DC}_X) = \{p \in (\mathbb{N}^\mathbb{N}) : S_p \subseteq X^{<\mathbb{N}} \text{ is a tree with } [S_p] \neq \emptyset\},$$
$$\Sigma^1_1\text{-}\mathsf{DC}_X(p) = [S_p]$$

Note that this formulation is different from Σ^1_1-dependent choice on X in the context of second order arithmetic. Indeed, our formulation falls between Σ^1_1-dependent choice and *strong* Σ^1_1*-dependent choice* (cf. Simpson [19, Definition VII.6.1]). Now, it is easy to see the following:

Proposition 2.2. $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$.

Proof. We prove the following Weihrauch inequality, in their order from left to rigth: $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}^\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}^\mathbb{N}}$.

$\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}^\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$: The set of all solutions to an instance of $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}^\mathbb{N}}$ is obviously Σ^1_1 relative to the given parameter, and one can easily find its Σ^1_1-index.

$\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathsf{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$: Let $P \subseteq \mathbb{N}^\mathbb{N}$ be a Σ^1_1 set. From its index, one can compute the index of a closed set $Q \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ such that the first projection of Q is P. Thus, given an element of Q, one can find an element of P by applying the first projection, and $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \geq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$: Given a code q of a closed subset C_q of $\mathbb{N}^\mathbb{N}$, compute a code of a pruned Σ^1_1 tree T such that $C_q = [T]$ (which is given by $\sigma \in T$ iff C_q contains an extension of σ). Then, let $\varphi_T(\sigma,\tau)$ be the formula expressing that τ is an immediate successor of σ in T. Moreover, $\langle p, q, \varepsilon\rangle$ where $S_q = \mathbb{N}^{<\mathbb{N}}$ and $S_p = \{\langle\sigma,\tau\rangle : \varphi_T(\sigma,\tau)\}$ satisfies the premise of $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ since T is pruned. Let f be a solution to this instance of $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$. Since T is pruned, f must be a path through T.

$\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}^\mathbb{N}}$: This is obvious as $\mathbb{N}$ can be computably embedded in $\mathbb{N}^\mathbb{N}$. □

2.2. **Compact Choice Principles.** According to the Arsenin-Kunugui uniformization theorem (cf. [13, Theorem 18.10]), the choice principle for σ-compact Δ^1_1 sets is much simpler than the one for arbitrary Δ^1_1 sets. We are interested in whether an analogous statement holds for Σ^1_1-choice, while we know that even compact Σ^1_1-choice does not admit a Borel uniformization.

We now consider subprinciples of the Σ^1_1 choice principle by restricting its domain. Recall that S_p is the analytic set in X coded by $p \in \mathbb{N}^\mathbb{N}$. Let $\mathcal{R}$ be a collection of subsets of X. Define $\Sigma^1_1\text{-}\mathsf{C}_X \upharpoonright_{\mathcal{R}}$, the Σ^1_1*-choice principle restricted to sets in* $\mathcal{R}$, as follows:

$$\begin{aligned}
\Sigma^1_1\text{-}\mathsf{C}_X \upharpoonright_{\mathcal{R}} &:\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows X,\\
\mathrm{dom}(\Sigma^1_1\text{-}\mathsf{C}_X \upharpoonright_{\mathcal{R}}) &= \{p \in \mathbb{N}^\mathbb{N} : S_p \neq \emptyset \text{ and } S_p \in \mathcal{R}\},\\
\Sigma^1_1\text{-}\mathsf{C}_X \upharpoonright_{\mathcal{R}} (p) &= S_p
\end{aligned}$$

First, we consider the Σ^1_1 choice principle restricted to compact sets, that is, we define *compact* Σ^1_1*-choice*, $\Sigma^1_1\text{-}\mathsf{KC}_X$, as follows:

$$\Sigma^1_1\text{-}\mathsf{KC}_X = \Sigma^1_1\text{-}\mathsf{C}_X \upharpoonright_{\{A\subseteq X : A \text{ is compact}\}}.$$

In other words, the Σ^1_1-compact choice principle, $\Sigma^1_1\text{-}\mathsf{KC}_X$, is the multivalued function which, given a code of a nonempty Σ^1_1 set which happens to be compact, chooses one element from the set. As the code contains no information about compactness, the principle of compact Σ^1_1 choice on $\mathbb{N}^\mathbb{N}$ should be considered as a Σ^1_1-version of König's lemma rather than of weak König's lemma. In contrast, a Σ^1_1-version of weak König's lemma and related principles are studied in [14]; e.g.,

- The principle $\Sigma^1_1\text{-}\mathsf{WKL}$, the *weak König's lemma for* Σ^1_1*-trees*, is the partial multivalued function which, given a Σ^1_1-name of a binary tree $T \subseteq 2^{<\mathbb{N}}$, chooses an infinite path through T.

- The principle Π^1_1-Sep, the *problem of separating a disjoint pair of Π^1_1 sets*, is the partial multivalued function which, given a Π^1_1-name of a pair of disjoint sets $A, B \subseteq \mathbb{N}$, chooses (the characteristic function of) a set $C \subseteq \mathbb{N}$ separating A from B, that is, $A \subseteq C$ and $B \cap C = \emptyset$.

Here, recall that p is called a Σ^1_1-name of S_p, and a Π^1_1-name of its complement.

Observation 2.3. Σ^1_1-$\mathsf{DC}_2 \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{WKL}$. □

Note that in [14, Lemma 4.6] these principles are shown to be equivalent to the parallelization of two-valued Σ^1_1 choice:

Fact 2.4 (Kihara-Marcone-Pauly [14])**.** $\widehat{\Sigma^1_1\text{-}\mathsf{C}_2} \equiv_{\mathsf{W}} \Pi^1_1$-$\mathsf{Sep} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{WKL}$.

Although König's lemma and weak König's lemma are different in the computability theoretic context, these are equivalent modulo some arithmetical power. Using this observation, we now see that the Σ^1_1 versions of König's lemma and weak König's lemma are computably equivalent, and thus, the principles mentioned in 2.4 are equivalent to Σ^1_1-compact choice.

Proposition 2.5. Σ^1_1-$\mathsf{KC}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{AC}_{\mathbb{N}\to 2} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{WKL} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{DC}_2$.

Proof. By Observation 2.1, we have $\widehat{\Sigma^1_1\text{-}\mathsf{C}_2} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{AC}_{\mathbb{N}\to 2}$. By Fact 2.4, Σ^1_1-$\mathsf{AC}_{\mathbb{N}\to 2} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{WKL}$. By Observation 2.3, we have Σ^1_1-$\mathsf{WKL} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{DC}_2$.

It remains to show that these are equivalent to the Σ^1_1 compact choice principle. First note that the reduction Σ^1_1-$\mathsf{WKL} \leq_{\mathsf{W}} \Sigma^1_1$-$\mathsf{KC}_{\mathbb{N}^{\mathbb{N}}}$ is obvious, as if $T \subseteq 2^{\mathbb{N}}$ is Σ^1_1, then $[T]$ is compact and Σ^1_1. For the converse, we claim that if a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^1_1 and compact then it is arithmetically isomorphic to a closed set $B \subseteq 2^{\mathbb{N}}$.

So suppose that A is Σ^1_1 and compact. First, it is clearly closed, so let $T_A \subseteq \mathbb{N}^{<\mathbb{N}}$ be a Σ^1_1 tree such that $A = [T_A]$ and T_A has no dead-end; that is, T_A is defined as $\sigma \in T_A$ iff $\exists X \succ \sigma$ with $X \in A$. By compactness and the fact that the tree is pruned, for every $\sigma \in \mathbb{N}^{<\mathbb{N}}$, there exists finitely many $i \in \mathbb{N}$ such that $\sigma^\frown i \in T_A$. We define T_B by $\forall \sigma \in 2^{<\mathbb{N}}$, $\sigma \in T_B$ if and only if $\exists \tau \in T_A$ such that $\sigma \prec 0^{\tau(0)}10^{\tau(1)} \cdots 0^{\tau(|\tau|-1)}1$. Then, the transformation $T_B \mapsto T_A$ is computable, while $T_A \mapsto T_B$ is T'_A-computable; hence $B = [T_B]$ is arithmetically isomorphic to A.

In particular, B is Σ^1_1, and moreover, from the above explicit definition of T_B, one can effectively compute a Σ^1_1-code of B from a given Σ^1_1-code of A. It is easy to compute an element of A from a given element of B. This argument shows that Σ^1_1-$\mathsf{WKL} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{KC}_{\mathbb{N}^{\mathbb{N}}}$. □

Next, we show that the compact Σ^1_1-choice principle is also Weihrauch equivalent to the following principles:

- The principle Π^1_1-Tot_2, the *totalization problem for partial Π^1_1 two-valued functions*, is the partial multivalued function which, given a Π^1_1-name of (the graph of) a partial function $\varphi\colon\subseteq \mathbb{N} \to 2$, chooses a total extension $f\colon \mathbb{N} \to 2$ of φ.
- The principle Π^1_1-DNC_2, the *problem of finding a two-valued diagonally non-Π^1_1 function*, is the partial multivalued function which, given a Π^1_1-name of a sequence of (the graphs of) partial functions $(\varphi_e)_{e\in\mathbb{N}}$, chooses a total function $f\colon \mathbb{N} \to 2$ diagonalizing the sequence, that is, $f(e) \neq \varphi_e(e)$ whenever $\varphi_e(e)$ is defined.

The latter notion has also been studied by Kihara-Marcone-Pauly [14, Section 4.2]. The former notion is easily seen equivalent to Π^1_1-Sep, which has also been studied in [14], with the advantage of being generalizable to $\mathbb{N}$. The following proposition is a consequence of [14, Theorem 4.3, Proposition 4.5] and Observation 2.1:

Proposition 2.6. Σ^1_1-$\mathsf{AC}_{\mathbb{N}\to 2} \equiv_{\mathsf{W}} \Pi^1_1$-$\mathsf{Tot}_2 \equiv_{\mathsf{W}} \Pi^1_1$-$\mathsf{DNC}_2$.

A set is *σ-compact* if it is a countable union of compact sets. By Saint Raymond's theorem (cf. [13, Theorem 35.46]), any Borel set with σ-compact sections can be written as a countable union of Borel sets with compact sections. In particular, a Borel code for a σ-compact set S can be computably transformed into a sequence of Borel codes of compact sets whose union is S. However, there is no analogous result for analytic sets (cf. Steel [21]). Therefore, we do not introduce σ-compact Σ^1_1-choice as

$$\Sigma^1_1\text{-}\mathsf{C}_X \restriction_{\{A\subseteq X : A \text{ is } \sigma\text{-compact}\}}.$$

Instead, we directly code an analytic σ-compact set as a sequence of analytic codes of compact sets. In other words, the *σ-compact Σ^1_1-choice principle*, Σ^1_1-$\mathsf{K}_\sigma\mathsf{C}_X$, is the partial multivalued function which, given a Σ^1_1-name of a sequence $(S_n)_{n\in\mathbb{N}}$ of compact sets such that at least one is nonempty, chooses an element from $\bigcup_{n\in\mathbb{N}} S_n$. Equivalently (modulo Weihrauch equivalence), one can formalize Σ^1_1-$\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ as the compositional product Σ^1_1-$\mathsf{KC}_{\mathbb{N}^\mathbb{N}} \star \Sigma^1_1$-$\mathsf{C}_\mathbb{N}$:

Lemma 2.7. Σ^1_1-$\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1$-$\mathsf{KC}_{\mathbb{N}^\mathbb{N}} \star \Sigma^1_1$-$\mathsf{C}_\mathbb{N}$.

Proof. We start by proving Σ^1_1-$\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1$-$\mathsf{KC}_{\mathbb{N}^\mathbb{N}} \star \Sigma^1_1$-$\mathsf{C}_\mathbb{N}$. Given a sequence (S_n) of compact Σ^1_1 sets, first use Σ^1_1-$\mathsf{C}_\mathbb{N}$ to get n with $S_n \neq \emptyset$ (as the set $\{n \in \mathbb{N} : \exists x \in S_n\}$ is Σ^1_1), and then Σ^1_1-$\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$ to choose an element of S_n.

For the converse direction, let $((S_n)_{n\in\mathbb{N}}, P)$ be an instance of Σ^1_1-$\mathsf{KC}_{\mathbb{N}^\mathbb{N}} \star \Sigma^1_1$-$\mathsf{C}_\mathbb{N}$, that is, for any n S_n is a Σ^1_1 set which is non-empty and compact whenever $n \in P$. Let $\hat{S}_n$ be the Σ^1_1 set defined by $x \in \hat{S}_n$ if and only if $n \in P$ and $x \in S_n$. The collection $(\hat{S}_n)_{n\in\mathbb{N}}$ is an instance of Σ^1_1-$\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, and any solution for it is a solution for the initial instance $(S_n)_{n\in\mathbb{N}}, P$. ☐

2.3. **Restricted Choice Principles.** Next, we consider several variations of the axiom of choice for φ:

(1) The axiom of *unique choice*: If for any $a \in X$, $\{b : \varphi(a,b)\}$ is a singleton, then there is a choice function for φ, that is, $\exists f \forall a\, \varphi(a, f(a))$.
(2) The axiom of *finite choice*: If for any $a \in X$, $\{b : \varphi(a,b)\}$ is nonempty and finite, then there is a choice function for φ, that is, $\exists f \forall a\, \varphi(a, f(a))$.
(3) The axiom of *cofinite choice*: If for any $a \in X$, $\{b : \varphi(a,b)\}$ is nonempty and cofinite, then there is a choice function for φ.
(4) The axiom of *finite-or-cofinite choice*: If for any $a \in X$, $\{b : \varphi(a,b)\}$ is nonempty and either finite or cofinite, then there is a choice function for φ.
(5) The axiom of *all-or-finite choice*: If for any $a \in X$, $\{b : \varphi(a,b)\}$ is nonempty and is either equal to X, or finite, then there is a choice function for φ.
(6) The axiom of *all-or-unique choice*: If for any $a \in X$, $\{b : \varphi(a,b)\}$ is nonempty and is either equal to X, or a singleton, then there is a choice function for φ.
(7) The axiom of *total unique choice*: There is a function such that whenever $\{b : \varphi(a,b)\}$ is a singleton, for $a \in X$, we have $\varphi(a, f(a))$.

The last notion is a modification of a variant of hyperarithmetical axiom of choice introduced by Tanaka [22] in the context of second order arithmetic, where the original formulation is given as follows:

$$\exists Z \forall n\, [\exists ! X \varphi(n, X) \longrightarrow \varphi(n, Z_n)],$$

where φ is a Σ^1_1 formula. We will only consider the above principles restricted to countable domains; that is, the principles of countable choice.

2.3.1. *Restricted single choice.* As in Observation 2.1, we may interpret these axioms of countable choice as parallelization of partial multivalued functions. To do so, we define:

$$\begin{aligned}
\Sigma^1_1\text{-}\mathsf{UC}_X &= \Sigma^1_1\text{-}\mathsf{C}_X \restriction_{\{A \subseteq X : |A|=1\}},\\
\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{fin}}_X &= \Sigma^1_1\text{-}\mathsf{C}_X \restriction_{\{A \subseteq X : A \text{ is finite}\}},\\
\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{cof}}_X &= \Sigma^1_1\text{-}\mathsf{C}_X \restriction_{\{A \subseteq X : A \text{ is cofinite}\}},\\
\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{foc}}_X &= \Sigma^1_1\text{-}\mathsf{C}_X \restriction_{\{A \subseteq X : A \text{ is finite or cofinite}\}},\\
\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aof}}_X &= \Sigma^1_1\text{-}\mathsf{C}_X \restriction_{\{A \subseteq X : A = X \text{ or } A \text{ is finite}\}},\\
\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_X &= \Sigma^1_1\text{-}\mathsf{C}_X \restriction_{\{A \subseteq X : A = X \text{ or } |A|=1\}}.
\end{aligned}$$

Note that the all-or-unique choice is often denoted by AoUC_X instead of $\mathsf{C}^{\mathsf{aou}}_X$, cf. [15]. In order to interpret the axiom of total unique choice as a multivalued function, we introduce the *totalization of the* Σ^1_1*-choice principle (restricted to* $\mathcal{R}$*) on* X. Recall that S_p is the analytic set in X coded by $p \in \mathbb{N}^{\mathbb{N}}$. Then we define $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{tot}}_X \restriction_{\mathcal{R}}$ as follows:

$$\begin{aligned}
\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{tot}}_X \restriction_{\mathcal{R}} &: \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N},\\
\mathrm{dom}(\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{tot}}_X \restriction_{\mathcal{R}}) &= \mathbb{N}^{\mathbb{N}},\\
\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{tot}}_X \restriction_{\mathcal{R}} (p) &= \begin{cases} S_p & \text{if } S_p \in \mathcal{R},\\ X & \text{otherwise.}\end{cases}
\end{aligned}$$

Roughly speaking, if a given Σ^1_1 set S is nonempty and belongs to $\mathcal{R}$, then any element of S is a solution to this problem as a usual choice problem, but even if a set S is either empty or does not belong to $\mathcal{R}$, there is a need to feed some value, although any value is acceptable as a solution.

In second order arithmetic, the totalization of dependent choice is known as *strong dependent choice* (cf. Simpson [19, Definition VII.6.1]). Here we consider the totalization $\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_X$ of $\Sigma^1_1\text{-}\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$, which can be viewed as the multivalued version of the axiom of *total unique choice* mentioned above:

$$\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_X = \Sigma^1_1\text{-}\mathsf{C}^{\mathsf{tot}}_X \restriction_{\{S \subseteq \mathbb{N}^{\mathbb{N}} : |S|=1\}}$$

2.3.2. *Restricted countable choice.* Hereafter, we will consider several restrictions of Σ^1_1 countable choice and Σ^1_1 dependent choice for numbers. Recall from Observation 2.1 that $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ can be identified with the parallelization of $\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}}$, and from Proposition 2.2 that $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ can be identified with $\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$.

Definition 2.8. We define several versions of axiom of choice where the set we have to choose from are restricted to special kinds:

$$\Sigma^1_1\text{-}\mathsf{AC}^{\triangledown}_{\mathbb{N}\to\mathbb{N}} = \widehat{\Sigma^1_1\text{-}\mathsf{C}^{\triangledown}_{\mathbb{N}}}$$

where $\triangledown \in \{\mathsf{fin}, \mathsf{cof}, \mathsf{foc}, \mathsf{aof}, \mathsf{aou}\}$ respectively corresponding to "finite", "cofinite", "finite or cofinite", "all or finite" and "all or unique". We will also consider the Dependent Choice with the same restricted sets:

$$\Sigma^1_1\text{-}\mathsf{DC}^{\triangledown}_{\mathbb{N}} = \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \restriction_{\{[T]:\forall\sigma\in T,\{n:\sigma^\frown n\in T\} \text{ is } \triangledown\}} .$$

where $\triangledown \in \{\mathsf{fin}, \mathsf{cof}, \mathsf{foc}, \mathsf{aof}, \mathsf{aou}\}$ has the same meaning. For any $\sigma \in \mathbb{N}^{<\mathbb{N}}$ a string corresponding to a choice for the previous sets, $\{n : \sigma^\frown n \in T\}$ corresponds to the next possible choice, and this set has to satisfy the condition specified by $\triangledown$. Note that it corresponds to a particular formulation of Σ^1_1 dependent choice, as explained just before Proposition 2.2.

In the following, we will say that a tree T is homogeneous if the set $[T]$ of all infinite paths through T is equal to some $\prod_{n\in\mathbb{N}} A_n$, that is $[T]$ is truly an instance of the axiom of choice. In other words, a homogeneous tree T is a tree where the set $\{n \in \mathbb{N} : \sigma^\frown n \in T\}$ depends only on $|\sigma|$ when $\sigma \in T$. Note that if T is homogeneous then the set $H := [T]$ satisfies the following property: We have $h \in H$ iff $\forall i \in \mathbb{N}$, $\exists f \in H$ with $h(i) = f(i)$.

2.3.3. *Weihrauch equivalences.* Among others, we see that "all-or-unique" choice is quite robust. Recall from Proposition 2.6 that the Π^1_1-totalization principle $\Pi^1_1\text{-}\mathsf{Tot}_2$ and the Π^1_1-diagonalization principle $\Pi^1_1\text{-}\mathsf{DNC}_2$ restricted to *two valued functions* are equivalent to the Σ^1_1 compact choice principle. We now consider the $\mathbb{N}$-valued versions of the totalization and the diagonalization principles:

- The principle $\Pi^1_1\text{-}\mathsf{Tot}_{\mathbb{N}}$, the *totalization problem for partial Π^1_1 functions*, is the partial multivalued function which, given a Π^1_1-name of (the graph of) a partial function $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$, chooses a total extension of φ.
- The principle $\Pi^1_1\text{-}\mathsf{DNC}_{\mathbb{N}}$, the *problem of finding a diagonally non-Π^1_1 function*, is the partial multivalued function which, given a Π^1_1-name of a sequence of (the graphs of) partial functions $(\varphi_e)_{e\in\mathbb{N}}$, chooses a total function $f : \mathbb{N} \to \mathbb{N}$ diagonalizing the sequence, that is, $f(e) \neq \varphi_e(e)$ whenever $\varphi_e(e)$ is defined.

It is clear that $\Pi^1_1\text{-}\mathsf{DNC}_{\mathbb{N}} \leq_{\mathsf{W}} \Pi^1_1\text{-}\mathsf{DNC}_2 \equiv_{\mathsf{W}} \Pi^1_1\text{-}\mathsf{Tot}_2 \leq_W \Pi^1_1\text{-}\mathsf{Tot}_{\mathbb{N}}$. One can easily see the following.

Proposition 2.9. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \Pi^1_1\text{-}\mathsf{Tot}_{\mathbb{N}}$.

Proof. The argument is almost the same as Proposition 2.6. Given a Π^1_1-name of a partial function φ, define $S_n = \{a : \varphi(n)\downarrow \to\ a = \varphi(n)\}$, which is uniformly Σ^1_1 (relative to the given name). Clearly, either $S_n = \mathbb{N}$ or S_n is a singleton. Hence, the all-or-unique choice principle chooses an element of S_n, which produces a totalization of φ.

Conversely, we first claim that for a Σ^1_1 set $S = \bigcap\{S_\alpha : \alpha < \omega_1^{\mathrm{CK}}\}$ (which is induced from a given Π^1_1 norm as explained in Section 1.2), if S is a singleton, say $S = \{n\}$, then there is $\alpha < \omega_1^{\mathrm{CK}}$ such that $S_\alpha = \{n\}$. This is because, for any $m \neq n$, we have $m \notin S = \bigcap_\alpha S_\alpha$, so there is a smallest $\alpha(m) < \omega_1^{\mathrm{CK}}$ such that $m \notin S_{\alpha(m)}$. As α is a Δ^1_1 function, by Spector bounding (Theorem 1.4), we must have $\alpha = \sup_{m\neq n} \alpha(m) < \omega_1^{\mathrm{CK}}$. Then $S_\alpha = \{n\}$, which verifies the claim.

Now, for the n-th Σ^1_1 set $S_n \subseteq \mathbb{N}$ with a Δ^1_1-approximation $(S_{n,\alpha})_{\alpha<\omega_1^{\mathrm{CK}}}$, we define a Δ^1_1 sequence (f_α) of partial functions on $\mathbb{N}$. First, let f_0 be the empty function, and then wait until $S_{n,\alpha}$ becomes a singleton (which is a Δ^1_1 property as

$S_{n,\alpha}$ is Δ^1_1) at some stage $\alpha < \omega_1^{\mathrm{CK}}$, say $S_{n,\alpha} = \{s_n\}$. If it happens, we define $f_\alpha(n) \downarrow= s_n$; otherwise $f_\alpha(n) \uparrow$. We eventually obtain (an index of) a partial function f with a Π^1_1 graph (i.e., a partial Π^1_1 function), which satisfies $f(n) = s_n$ whenever $S_n = \{s_n\}$ by the above claim. In particular, for any total extension $\widehat{f}$ of f, we have $\widehat{f}(n) \in S_n$ whenever $S_n = \mathbb{N}$ or S_n is a singleton.

Now, the construction (i.e., the inner reduction) from S to f is clearly uniform, and relativizable (see also the argument in Section 1.2). The outer reduction is trivial. Therefore, these give a Weihrauch reduction. □

We now show that all-or-unique choice is also Weihrauch equivalent to total unique choice.

Proposition 2.10. *Let X be a Δ^1_1 subset of $\mathbb{N}$. Then, $\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_X \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_X$.*

Proof. We start by proving that $\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_X \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_X$. Consider the following computable inner reduction, which given an index for a Σ^1_1 set $S = \bigcap\{S_\alpha : \alpha < \omega_1^{\mathrm{CK}}\}$, output an index for the following Σ^1_1 set $R \subseteq X$ (that we describe as a co-enumeration along ω_1^{CK}): First R does nothing until an ordinal stage where (the co-enumeration of) S is a singleton; that is, for each stage $\alpha < \omega_1^{\mathrm{CK}}$, check if the Δ^1_1 set S_α is a singleton or not (which is a Δ^1_1 property). If it happens, R removes all integers so that $R_\alpha = S_\alpha$; if not, we keep $R_\alpha = X$. Since either $R = X$ or R is a singleton, R is an instance of $\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_X$. By the claim in the proof of Proposition 2.9, if S is a singleton then it is witnessed at some stage before ω_1^{CK} and so $R = S$; therefore, the identity map trivially gives an outer reduction. Moreover, the construction is effective; that is, given a Σ^1_1-code of S, one can effectively find a Σ^1_1-index of R.

We now prove that $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_X \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_X$. Every instance of $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_X$ is an instance of $\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_X$ with the same solution, so this is trivial. □

In particular, the totalization of two-valued unique choice is equivalent to the compact choice.

Corollary 2.11. $\widehat{\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_2} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$.

Proof. It is clear that $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_2 \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_2$, so $\widehat{\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_2} \equiv_{\mathsf{W}} \widehat{\Sigma^1_1\text{-}\mathsf{C}_2} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to 2}$ by Observation 2.1. Thus, the assertion follows from Fact 2.4 and Proposition 2.5 □

2.4. **Arithmetical Transfinite Recursion.** In reverse mathematics, the axiom of Σ^1_1-choice is known to be weaker than the arithmetical transfinite recursion scheme ATR_0 (cf. [19, Section VIII.4]). However, an analogous result does not hold in the Weihrauch context. The purpose of this section is to clarify the relationship between the Σ^1_1-choice principles and the arithmetical transfinite recursion principle in the Weihrauch lattice.

Kihara-Marcone-Pauly [14] first introduced an analogue of *arithmetical transfinite recursion*, ATR_0, in the context of Weihrauch degrees, and studied two-sided versions of several dichotomy theorems related to ATR_0, but they have only considered the one-sided version of ATR_0. Then, Goh [9] introduced the two-sided version of ATR_0 to examine the Weihrauch strength of König's duality theorem for infinite bipartite graphs. Roughly speaking, the above two Weihrauch problems are introduced as follows:

- The *one-sided version*, ATR, by [14] is the partial multivalued function which, given a countable well-ordering $\prec$, returns the jump hierarchy for $\prec$.
- The *two-sided version*, ATR_2, by [9] is the total multivalued function which, given a countable linear ordering $\prec$, chooses either a jump hierarchy for $\prec$ or an infinite $\prec$-decreasing sequence.

Here, a *jump hierarchy* for a partially ordered set $(P, <_P)$ is a sequence $(H_p)_{p\in P}$ of sets satisfying the following property: For all $p \in P$,

$$H_p = \bigoplus_{q<_P p} H'_q,$$

where $\bigoplus_n S_n$ denotes the usual Turing join (i.e., the *coproduct*) defined by $\bigoplus_n S_n = \{\langle n, x\rangle : x \in S_n\}$, and H' denotes the Turing jump of H. Note that the definition of a jump hierarchy is clearly described by an arithmetical condition. For more details, see also Sacks [18, Section II.4].

Even if $\prec$ is not well-founded, some solution to $\mathsf{ATR}_2(\prec)$ may produce a jump hierarchy for $\prec$ (often called a *pseudo-hierarchy*) by Harrison's well-known result that there exists pseudo-well-orders which admit a jump hierarchy (but in this case the jump hierarchy is not necessarily unique). Regarding ATR_2, we note that, sometimes in practice, what we need is not a full jump hierarchy for a pseudo-well-ordering, but a jump hierarchy for an initial segment of $\prec$ containing its well-founded part. Therefore, we introduce another two-sided version $\mathsf{ATR}_{2'}$ as follows:

Let L be a linearly ordered set. The *well-founded part* of L is the largest initial segment of L which is well-founded. We say that an initial segment I of L is *large* if it contains the well-founded part of L.

We consider a variant of the arithmetical transfinite recursion $\mathsf{ATR}_{2'}$, which states that for any linear order $\prec_x$ coded by x, one can find either a jump hierarchy for a large initial segment of $\prec_x$ or an infinite $\prec_x$-decreasing sequence:

$$\begin{aligned}\mathsf{ATR}_{2'}(x) = \{&0^\frown H : H \text{ is a jump hierarchy for a large initial segment of } \prec_x\}\\ &\cup \{1^\frown p : p \text{ is an infinite decreasing sequence with respect to } \prec_x\}.\end{aligned}$$

Seemingly, $\mathsf{ATR}_{2'}$ is completely unrelated to any other choice principles. Surprisingly, however, we will see that the parallelization of $\mathsf{ATR}_{2'}$ is arithmetically equivalent to the choice principle for Σ^1_1-compact sets. We say that f is *arithmetically Weihrauch reducible to* g (written $f \leq^a_{\mathsf{W}} g$) if we are allowed to use arithmetic functions H and K (i.e., $H, K \leq_{\mathsf{W}} \lim^{(n)}$ for some $n \in \mathbb{N}$) in the definition of Weihrauch reducibility.

Theorem 2.12. $\widehat{\mathsf{ATR}_{2'}} \equiv^a_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}} \equiv^a_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$.

We divide the proof of Theorem 2.12 into two lemmas.

Lemma 2.13. $\mathsf{ATR}_{2'} \leq^a_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$.

Proof. By Corollary 2.11, it suffices to prove $\mathsf{ATR}_{2'} \leq^a_{\mathsf{W}} \widehat{\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_2}$. Fix x a code for a linear order. Given $n \in \mathbb{N}$, let JH_n be the set of jump hierarchies for $\prec_x\restriction_n$ (where $\prec_x\restriction_n$ is the restriction of $\prec_x$ up to $\{y : y \prec_x n\}$). Note that $\{(n, H) : H \in JH_n\}$ is arithmetical (since the definition of a jump hierarchy is described by an arithmetical condition as mentioned above). For $a, k \in \mathbb{N}$, if $a \prec_x n$ then we consider the set $S^n_{a,k}$ of all possible values of $H_a(k)$, the k-th value of the a-th level of H, for some jump hierarchy $H \in JH_n$, i.e., $S^n_{a,k} = \{H_a(k) : H \in JH_n\}$. If $a \not\prec_x n$ then put $S^n_{a,k} = \{0\}$. Clearly, $S^n_{a,k}$ is Σ^1_1 uniformly in n, a, k, and therefore there is a computable function f such that $S^n_{a,k}$ is the $f(n,a,k)$-th Σ^1_1 set. Note that if $\prec_x\restriction_n$ is well-founded, then the product $S^n := \prod_{\langle a,k\rangle} S^n_{a,k}$ essentially[1] consists of the unique jump hierarchy for $\prec_x\restriction_n$. In particular, $S^n_{a,k}$ is a singleton for any $a \prec_x n$ and $k \in \mathbb{N}$ whenever $\prec_x\restriction_n$ is well-founded.

Given $p_{n,a,k} \in \Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_2(f(n,a,k))$ (that is, $p_{n,a,k} \in S^n_{a,k}$ whenever $S^n_{a,k}$ is a singleton), define $H_n = \bigoplus_{\{\langle a,k\rangle : a \prec_x n\}} p_{n,a,k}$. Note that if n is contained in the well-founded part of $\prec_x$, then H_n must be the jump hierarchy for $\prec_x\restriction_n$. By using arithmetical power (being a jump hierarchy is a Π^0_2 statement), first ask if H_n is a jump hierarchy for $\prec_x\restriction_n$ for every n. If yes, $\bigoplus_n H_n$ is a jump hierarchy along the whole ordering $\prec_x$. In particular, the whole ordering includes the well-founded part of $\prec_x$, and therefore is large.

If no, we claim that there is no $\prec_x$-least n such that H_n is not a jump hierarchy for $\prec_x\restriction_n$. Indeed, suppose there exists one, then obviously n is not contained in the well-founded part of $\prec_x$. Hence, $\prec_x\restriction_n$ is a large initial segment of $\prec_x$. Moreover, by minimality of n, every H_j for $j \prec_x n$ is a jump hierarchy, so by definition of a jump hierarchy, $\bigoplus\{H'_j : j \prec_x n\}$ is the jump hierarchy for $\prec_x\restriction_n$. But then, none of the $S^n_{a,k}$ are empty, and H_n must be a jump hierarchy for $\prec_x\restriction_n$, a contradiction proving the claim.

Let j_0 be the $<_{\mathbb{N}}$-least number such that H_{j_0} is not a jump hierarchy for $\prec_x\restriction_{j_0}$, and $j_{n+1} \prec_x j_n$ be the $<_{\mathbb{N}}$-least number such that $H_{j_{n+1}}$ is not a jump hierarchy for $\prec_x\restriction_{j_{n+1}}$. Finding such a sequence is Δ^0_3, so one can arithmetically find such an infinite sequence $(j_n)_{n\in\mathbb{N}}$, which is clearly decreasing with respect to $\prec_x$. $\square$

Lemma 2.14. $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aof}}_{\mathbb{N}} \leq^a_{\mathsf{W}} \widehat{\mathsf{ATR}_{2'}}$.

Proof. First, consider a computable instance S of $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aof}}_{\mathbb{N}}$. Let $\prec_n$ be a linear order on an initial segment L_n of $\mathbb{N}$ such that $n \in S$ iff $\prec_n$ is ill-founded. Let H_n be a solution to the instance $\prec_n$ of $\mathsf{ATR}_{2'}$. Ask the arithmetic question whether there exists n such that H_n is an infinite decreasing sequence w.r.t. $\prec_n$. If so, one can computably find such an n, which belongs to S. Otherwise, each H_n is (essentially) a jump hierarchy along a large initial segment J_n of L_n. In an arithmetical way, one can obtain J_n. Then ask if $L_n \setminus J_n$ is nonempty, and has no $\prec_n$-minimal element. If the answer to this arithmetical question is yes, we have $n \in S$.

Thus, we are left with the case where for any n either $L_n = J_n$ holds or $L_n \setminus J_n$ has a $\prec_n$-minimal element. In this case, if $n \in S$ then J_n is ill-founded. This is because if J_n is well-founded, then J_n is exactly the well-founded part of L_n since J_n is large, and thus $L_n \setminus J_n$ is nonempty and has no $\prec_n$-minimal element. Moreover, since J_n admits a jump hierarchy while it is ill-founded, J_n is a pseudo-well-order; hence H_n computes all hyperarithmetical reals (see [6]). Conversely, if $n \notin S$ then H_n is a jump hierarchy along the well-order $J_n = L_n$, which is hyperarithmetic.

Now, ask if the following $(H_n)_{n\in\mathbb{N}}$-arithmetical condition holds:

$$(\exists i)(\forall j)\ H_i \not<_T H_j. \tag{1}$$

By our assumption that $S \neq \emptyset$, there is $k \in S$, so that H_k computes all hyperarithmetic reals. Therefore, if (1) is true with witness i, the hierarchy H_i cannot

[1] It coincides in the domain of $\prec_x\restriction_n$, however for $a \not\prec_x n$, we have $S^n_{a,k} = \{0\}$.

be hyperarithmetic; hence $i \in S$. Then one can arithmetically find such an i. If (1) is false, for any i there is j such that $H_i <_T H_j$. In this case, start from $i := k \in S$, and obtain an infinite sequence $j_0, j_1, j_2, \dots$ such that $H_k <_T H_{j_0} <_T H_{j_1} <_T \dots$. Since H_k computes all hyperarithmetical sets, H_{j_n} is not hyperarithmetical for any n, i.e., $j_n \in S$. This implies that S is an infinite set. However, by our assumption, if S is infinite, then $S = \mathbb{N}$. Hence, any i is solution to S.

Finally, one can uniformly relativize this argument to any instance of $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aof}}_{\mathbb{N}}$. □

Proof of Theorem 2.12. By Lemma 2.13, $\widehat{\mathsf{ATR}_{2'}} \leq^a_{\mathsf{W}} \widehat{\Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}}$. By Corollary 2.11, $\widehat{\Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}} \equiv_{\mathsf{W}} \widehat{\widehat{\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_2}} \equiv_{\mathsf{W}} \widehat{\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_2} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$. By Proposition 2.10 and Definition 2.8, $\widehat{\Sigma^1_1\text{-}\mathsf{UC}^{\mathsf{tot}}_2} \equiv_{\mathsf{W}} \widehat{\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aou}}_X} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$.

Clearly, $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$, as witnessed by the identity reductions. By Definition 2.8 and Lemma 2.14, $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \widehat{\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aof}}_{\mathbb{N}}} \leq^a_{\mathsf{W}} \widehat{\widehat{\mathsf{ATR}_{2'}}} \equiv_{\mathsf{W}} \widehat{\mathsf{ATR}_{2'}}$. □

One can also consider a jump hierarchy for a partial ordering. Then, we consider the following partial order version of Goh's arithmetical transfinite recursion. Let $(\prec_x)$ be a coding system of all countable partial orderings.

$$\begin{aligned}\mathsf{ATR}^{\mathsf{po}}_2(x) = \{&0^\frown H : H \text{ is a jump hierarchy for } \prec_x\}\\ &\cup \{1^\frown p : p \text{ is an infinite decreasing sequence with respect to } \prec_x\}.\end{aligned}$$

Note that $\mathsf{ATR}^{\mathsf{po}}_2(x)$ is an arithmetical subset of $\mathbb{N}^\mathbb{N}$. Obviously,

$$\mathsf{ATR} \leq_{\mathsf{W}} \mathsf{ATR}_{2'} \leq_{\mathsf{W}} \mathsf{ATR}_2 \leq_{\mathsf{W}} \mathsf{ATR}^{\mathsf{po}}_2 \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}}.$$

This version of arithmetical transfinite recursion directly computes a solution to all-or-finite choice on the natural numbers without using parallelization or arithmetical Weihrauch reductions.

Proposition 2.15. $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aof}}_{\mathbb{N}} \leq_{\mathsf{W}} \mathsf{ATR}^{\mathsf{po}}_2$.

Proof. Let S be a computable instance of $\Sigma^1_1\text{-}\mathsf{C}^{\mathsf{aof}}_{\mathbb{N}}$. Let T_n be a computable tree such that $n \in S$ iff T_n is ill-founded. Define

$$T = 00 \sqcup_n T_n = \{\langle\rangle, \langle 0\rangle, \langle 00\rangle\} \cup \{\langle 00n\rangle\sigma : \sigma \in T_n\}.$$

Let $i^\frown H$ be a solution to the instance T (ordered by reverse inclusion) of $\mathsf{ATR}^{\mathsf{po}}_2$. If $i = 1$, i.e., if H is an infinite decreasing sequence w.r.t. T, then this provides an infinite path p through T. Then, choose n such that $00n \prec p$, which implies T_n is ill-founded, and thus $n \in S$. Otherwise, $i = 0$, and thus H is a jump hierarchy for T. We define $H^*_n = H_{\langle 00n\rangle}$. Note that if $n \notin S$ then H^*_n is hyperarithmetic, and if $n \in S$ then H^*_n computes all hyperarithmetical reals. By the definition of a jump hierarchy, we have $(H^*_n)'' \leq_T H$. Thus, the following is an H-computable question:

$$(\exists i)(\forall j)\ H^*_i \not<_T H^*_j. \tag{2}$$

As in the proof of Lemma 2.14, one can show that if (2) is witnessed by i then $i \in S$, and if (2) is false then any i is a solution to S. As before, one can uniformly relativize this argument to any instance S. □

Question 2.16. $\mathsf{ATR}_2 \equiv^a_{\mathsf{W}} \mathsf{ATR}_{2'} \equiv^a_{\mathsf{W}} \mathsf{ATR}^{\mathsf{po}}_2$?

3. An analysis of the analytic axioms of choice and dependent choice

In this section, we investigate the structure of different restrictions of the axiom of analytical choice under the Weihrauch reducibility. We compare the dependent and independent axiom of choice for the various restrictions, and the relative strength of the restrictions.

We show both Weihrauch reductions and non reductions. A powerful tool for proving the latter is Medvedev reduction, introduced in [16] to classify problems according to their degree of difficulty, as for Weihrauch reducibility. However, when Weihrauch reducibility compare problems that have several instances, each of them with multiple solutions, Medvedev reducibility compare "mass problems", which correspond to problems with a single instance. A mass problem is a set of functions from natural numbers to natural numbers, representing the set of solutions. For two mass problems $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$, we say that P is Medvedev reducible to Q if every solution for Q uniformly computes a solution for P.

Definition 3.1 (Medvedev reduction)**.** Let $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$ be sets. We say that P is Medvedev reducible to Q, written $P \leq_M Q$ if there exists a single computable function f such that for every $x \in Q$, $f(x) \in P$.

If $P :\subseteq X \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is now a Weihrauch problem, that is a partial multivalued function, then for any instance $x \in X$, one can consider the mass problem $P(x)$ (the set of all solutions of the x-th instance of P). Then, if $P \leq_W Q$, then for every computable instance x of P, there is a computable instance y of Q such that $P(x) \leq_M Q(y)$. Using Medvedev reducibility, we are able to compare the degree of complexity of different instances of the same problem, and we will be interested in the structural property of their complexity: Given a Weihrauch problem P, we define the Medvedev partial order of P to be the partial order of Medvedev degrees of $P(x)$ for all computable instances $x \in \mathrm{dom}(P)$, under the Weihrauch reduction. We will see below Definition 2.8 that when P corresponds to a restriction of the axiom of choice, the corresponding partial order is an upper semi-lattice, while when it corresponds to a restriction of the axiom of dependent choice, the corresponding partial order is a lattice. For instance, $P = \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ yields the *Medvedev lattice of* Σ^1_1*-closed sets*, which is interesting in its own right.

We will be mainly interested in the existence of maximal elements of Medvedev semi-lattices of P, for P being various choice problems, as it can be used to Weihrauch-separate two problems. Suppose that $P \leq_W Q$ and the Medvedev semi-lattice of Q has no maximal element, while the Medvedev semi-lattice of P has one. Then, we have $P <_W Q$: Let $x \in \mathrm{dom}(P)$ be any computable instance realizing a maximal Medvedev degree in P, and take $y \in \mathrm{dom}(Q)$ computable such that $P(x) \leq_M Q(y)$ (as $P \leq_W Q$). By the fact that Q has no maximal element, let $z \in \mathrm{dom}(Q)$ be computable and such that $Q(z) >_M Q(y)$. Then, it cannot be that there is $t \in \mathrm{dom}(P)$ computable such that $P(t) \geq_M Q(z)$, as it would contradict maximality of x. Therefore, z is a witness that $P <_W Q$.

Throughout this section, we use the following abuse of notation.

Notation. Given a Weihrauch problem P, we abuse notation by writing "$A \in P$" or "A in P" to mean that A is a computable instance of P, that is, $A = P(x)$ for some computable $x \in \mathrm{dom}(P)$.

Given $\triangledown \in \{\mathsf{fin}, \mathsf{cof}, \mathsf{foc}, \mathsf{aof}, \mathsf{aou}\}$, the partial order on the computable instances of $\Sigma^1_1\text{-}\mathsf{AC}^{\triangledown}_{\mathbb{N}\to\mathbb{N}}$ under Medvedev reducibility form an upper semi-lattice. Indeed, if

$A = \prod_n A_n$ and $B = \prod_n B_n$ are two computable instances of $\Sigma^1_1\text{-}\mathsf{AC}^{\triangledown}_{\mathbb{N}\to\mathbb{N}}$, then $C = \prod_n C_n$ where $C_{2n} = A_n$ and $C_{2n+1} = B_n$ is a computable instance of $\Sigma^1_1\text{-}\mathsf{AC}^{\triangledown}_{\mathbb{N}\to\mathbb{N}}$ which is a least upper bound of A and B. Similarly, the partial order on the computable instances of $\Sigma^1_1\text{-}\mathsf{DC}^{\triangledown}_{\mathbb{N}}$ under Medvedev reducibility form a lattice: If A and B are instances of $\Sigma^1_1\text{-}\mathsf{DC}^{\triangledown}_{\mathbb{N}}$, then $C = \{\tau \in \mathbb{N}^{<\mathbb{N}} : \sigma_0 : n \mapsto \tau(2n) \in A \wedge \sigma_1 : n \mapsto \tau(2n+1) \in B\}$ is a computable instance of $\Sigma^1_1\text{-}\mathsf{DC}^{\triangledown}_{\mathbb{N}}$ which is a least upper bound of A and B, and for $\triangledown \in \{\mathsf{fin}, \mathsf{foc}, \mathsf{aof}\}$ (respectively $\triangledown \in \{\mathsf{cof}, \mathsf{aou}\}$) the set $D = 0^\frown A \cup 1^\frown B$ (respectively $D = \bigcup_n (2n^\frown A) \cup (2n+1^\frown B))$ is a computable instance of $\Sigma^1_1\text{-}\mathsf{DC}^{\triangledown}_{\mathbb{N}}$ which is a greatest lower bound for A and B.

This structural difference imply that no matter the restriction on analytical AC and DC, the corresponding semi-lattices will always be different under Medvedev reducibility, as there exists two homogeneous sets whose product is not Medvedev equivalent to a homogeneous set.

Proposition 3.2. *For every $\triangledown \in \{\mathsf{fin}, \mathsf{cof}, \mathsf{foc}, \mathsf{aof}, \mathsf{aou}\}$, there exists $A \in \Sigma^1_1\text{-}\mathsf{DC}^{\triangledown}_{\mathbb{N}}$ such that there is no $B \in \Sigma^1_1\text{-}\mathsf{AC}^{\triangledown}_{\mathbb{N}\to\mathbb{N}}$ with $A \equiv_M B$.*

Proof. Simply take A_0 and A_1 in $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$ with are not Medvedev equivalent, and consider $C = 0^\frown A_0 \cup 1^\frown A_1$, which is in $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$. Now, toward a contradiction, suppose also that there exists H in $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$ (actually there is no need for H to be Σ^1_1) such that $C \equiv_M H$. Let ϕ and ψ be witness of this, i.e ϕ (resp. ψ) is total on C (resp. H) and its image is included in H (resp. C).

Now, we describe a way for some A_i to Medvedev compute A_{1-i}: Let $i \in 2$ and σ be extendible in H such that $\psi^\sigma(0) = 1 - i$. Given $x \in A_i$, apply ϕ on $i^\frown x$ to obtain an element y of H. Define $\widetilde{y}$ to be y with its beginning replaced by σ. Then, by homogeneity, $\widetilde{y}$ is still in H, so $\phi(\widetilde{y})$ has to be in $(1-i)^\frown A_{1-i}$.

For other values of $\triangledown$, the proof is exactly the same or very similar. $\square$

Note that the above proof used the fact that there always exists infimum in $\Sigma^1_1\text{-}\mathsf{DC}^{\triangledown}_{\mathbb{N}}$ while this is not clear in $\Sigma^1_1\text{-}\mathsf{AC}^{\triangledown}_{\mathbb{N}\to\mathbb{N}}$.

3.1. **Axioms of finite analytic choice.** We already have defined $\Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$ in Section 2.2, which is clearly the same problem as $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$ up to the coding of the instance. Using our previous work, we show that the finite choice can always be weakened to independent choice over 2 possibilities:

Theorem 3.3. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}} \equiv_W \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}} \equiv_W \Sigma^1_1\text{-}\mathsf{WKL}$.

Proof. First, it is clear that we have $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$. Up to the coding of the instance, we also have $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$. By Proposition 2.5, $\Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to 2} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{WKL}$. But clearly, $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to 2} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$. $\square$

In the following, we are interested in a finer analysis of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$ and $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$ using Medvedev reducibility. In particular, we show that both of these semi-lattices admit a maximal element: Indeed, we show that there is a single nonempty compact homogeneous Σ^1_1 set coding all information of nonempty compact Σ^1_1 sets. This can be viewed as an effective version of Dellacherie's theorem (cf. Steel [21]) in descriptive set theory.

Theorem 3.4. *There exists a maximum in the Medvedev semi-lattices of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$ and $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$.*

Proof. To construct a greatest element in Σ^1_1-$\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$, by Proposition 2.5, we only need to enumerate all nonempty compact Σ^1_1 subsets of $2^{\mathbb{N}}$. Let S_e be the e-th Σ^1_1 subtree of $2^{<\mathbb{N}}$. Then, consider a Δ^1_1 co-enumeration $(S_{e,\alpha})_{\alpha<\omega_1^{\mathrm{CK}}}$ of S_e (induced from a Π^1_1-norm, as explained in Section 1.2). If $[S_e]$ is empty, by compactness, S_e is finite.

If S_e is finite, we claim that there is $\alpha < \omega_1^{\mathrm{CK}}$ such that $S_{e,\alpha}$ is finite, and the least such α is a successor ordinal. To see this, first note that for each $\sigma \in 2^{<\mathbb{N}} \setminus S_e$, there is $\alpha(\sigma) < \omega_1^{\mathrm{CK}}$ such that $\sigma \notin S_{e,\alpha(\sigma)}$. Now, $2^{<\mathbb{N}} \setminus S_e$ is cofinite, and in particular, computable, so by Spector bounding (Theorem 1.4), we have $\alpha = \sup\{\alpha(\sigma) : \sigma \in 2^{<\mathbb{N}} \setminus S_e\} < \omega_1^{\mathrm{CK}}$. For the second assertion of our claim, by minimality of α, we have $[S_{e,\beta}] \neq \emptyset$ for any $\beta < \alpha$, and moreover, if α is a limit ordinal, then $[S_{e,\alpha}] = \bigcap [S_{e,\alpha[n]}]$. However, by compactness, this intersection is nonempty, which contradicts our choice of α.

Now we construct a uniform Δ^1_1 approximation of a sequence $(T_e)_{e\in\mathbb{N}}$ of nonempty Σ^1_1 sets such that if $[S_e] \neq \emptyset$ then $T_e = [S_e]$. Define $T_{e,0} = 2^{\mathbb{N}}$, and for any $\alpha > 0$, $T_{e,\alpha} = [S_{e,\alpha}]$ if $[S_{e,\alpha}] \neq \emptyset$ (which is a $\Pi^0_1(S_{e,\alpha})$ property by compactness, so in particular Δ^1_1). If $\alpha > 0$ is the first stage such that $[S_{e,\alpha}] = \emptyset$, then, by the above claim, α is a successor ordinal, say $\alpha = \beta + 1$. In this case, define $T_{e,\gamma} = T_{e,\beta}$ for any $\gamma \geq \alpha$, and end the construction. By minimality of α, we have $T_{e,\beta} \neq \emptyset$ since $\beta < \alpha$. Then consider $T_e = \bigcap_\alpha T_{e,\alpha}$, and it is not hard to check that the sequence $(T_e)_{e\in\mathbb{N}}$ has the desired property. It is easy to check that this argument is uniformly relativizable to any oracle.

As a maximal instance of Σ^1_1-$\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$, it suffices to take the one consisting of the product of all T_e. Note that by Theorem 3.3 this also shows the maximality result for Σ^1_1-$\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$. □

As a special property of Σ^1_1 compact sets, we have the following analog of the hyperimmune-free basis theorem. For $p, q \in \mathbb{N}^{\mathbb{N}}$ we say that p is *higher Turing reducible to* q (written $p \leq_{hT} q$) if there is a partial Π^1_1-continuous function $\Phi\colon\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\Phi(q) = p$ (see Bienvenu-Greenberg-Monin [1] for more details). Here, Φ is Π^1_1-continuous if $\Phi(q)(n)\downarrow = m$ is a Π^1_1 relation.

Lemma 3.5. *For any Σ^1_1 compact set $K \subseteq \mathbb{N}^{\mathbb{N}}$ there is an element $p \in K$ such that every $f \leq_{hT} p$ is majorized by a Δ^1_1 function.*

Proof. Let (ψ_e) be a list of higher Turing reductions. Let $K_0 = K$. For each e, let $Q_{e,n} = \{x \in \mathbb{N}^{\mathbb{N}} : \psi^x_e(n)\uparrow\}$. Then $Q_{e,n}$ is a Σ^1_1 closed set. If $K_e \cap Q_{e,n}$ is nonempty for some n, define $K_{e+1} = K_e \cap Q_{e,n}$ for such n; otherwise define $K_{e+1} = K_e$. Note that if $K_e \cap Q_{e,n}$ is nonempty for some n, then ψ^x_e is not total for any $x \in K_{e+1}$. If $K_e \cap Q_{e,n}$ is empty for all n, then ψ_e is total on the Σ^1_1 compact set K_e, one can find a Δ^1_1 function majorizing ψ^x_e for all $x \in K_e$ (cf. the proof of [14, Lemma 4.7]). Define $K_\infty = \bigcap_n K_n$, which is nonempty since K_n is compact. Then, for any $p \in K_\infty$, every $f \leq_{hT} p$ is majorized by a Δ^1_1 function. □

Note that continuity of higher Turing reduction is essential in the above proof. Indeed, one can show the following:

Proposition 3.6. *There is a nonempty Σ^1_1 compact set $K \subseteq \mathbb{N}^{\mathbb{N}}$ such that for any $p \in K$, there is $f \leq_T p'$ which dominates all Δ^1_1 functions.*

Proof. Let (φ_e) be an effective enumeration of all partial Π^1_1 functions $\varphi_e\colon\subseteq \mathbb{N} \to 2$. As in the argument in Proposition 2.6 or Proposition 2.9, one can see that the set

S_e of all two-valued totalizations of the partial Π^1_1 function φ_e is nonempty and Σ^1_1. Then the product $K = \prod_e S_e$ is also a nonempty Σ^1_1 subset of $2^{\mathbb{N}}$. It is clear that every $p \in K$ computes any two-valued total Δ^1_1 function, so (non-uniformly) computes any total Δ^1_1 function on $\mathbb{N}$. Let BB be a total p'-computable function which dominates all p-computable functions. In particular, $BB \leq_T p'$ dominates all Δ^1_1 functions. □

3.2. **Axioms of all-or-finite and all-or-unique analytic choice.** We now discuss choice when the sets from which we choose can be either everything, or finite. We will show that under the Weihrauch scope, this principle is a robust one, in the sense of having multiple characterization, that is strictly above $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$. It also share with the latter that dependent and independent choice are equivalent and the existence of a maximal element containing all the information, with very similar proof as for $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$.

In Proposition 2.9, we hinted that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$ is robust. We give two other evidences of this in the following theorems.

Theorem 3.7. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$

Proof. It is clear that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \geq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$, by the identity function. It remains to prove $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$. Let $A = \prod_n A_n \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$. We define uniformly in A the set $B = \prod_{\langle m,n\rangle} B^m_n$ such that $A \leq_M B$ and $B \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$. We will ensure that there exists a single computable function Φ such that for any m and $X \in \prod_n B^m_n$ we have $\Phi^X \in A_m$.

We first describe the co-enumeration of B^m_n. Let $(A_{m,\alpha})_{\alpha<\omega_1^{\mathrm{CK}}}$ be an approximation of $A_m \subseteq \mathbb{N}$ (induced from a Π^1_1-norm, as explained in Section 1.2). First, wait for the first stage where A_m is finite (if this never happens then $B^m_n = \mathbb{N}$). Here note that if A_m is finite then its finiteness is witnessed at some stage before ω_1^{CK} by Spector bounding (Theorem 1.4) as seen in the proof of Theorem 3.4. If it happens, wait for exactly n additional elements to be removed from A_m. If this happens, let $B^m_n = \{c\}$ where c is the code for the finite set A_m at this stage α_n. More formally, we suppose that in the co-enumeration of A_m at most one element is removed at each stage, and we let α_0 be the least stage such that A_{m,α_0} is finite, and α_{n+1} be the least stage such that $A_{m,\alpha_{n+1}} \subsetneq A_{m,\alpha_n}$ if it exists. Then, let D_e be the finite set coded by e, and set $B^m_n = \{c\}$ with $D_c = A_{m,\alpha_n}$.

Now, we describe the function Φ. Given X, find the first i such that we do not have the following: $D_{X(i)} \supsetneq D_{X(i+1)} \neq \emptyset$. Note that $X(0)$ codes a finite set, so the length of the chain $D_{X(0)} \supsetneq D_{X(1)} \supsetneq \dots$ has to be finite. Therefore, there exists such an i. Then, output any element Φ^X from $D_{X(i)}$. if $X \in \prod_n B^m_n$, whenever we reach stage α_n, we have $D_{X(n)} = A_{m,\alpha_n}$, and thus $i \geq n$. Assume that α_k is the last stage such that some element is removed from A_m. Then, $k \leq i$ and $D_{X(k)} = A_{m,\alpha_k} = A_m \supseteq D_{X(i)}$. This implies that if $X \in \prod_n B^m_n$, then the chosen element Φ^X is contained in A_m, as required. □

We have seen by combining Proposition 2.6, Theorem 3.3 and Proposition 2.5 that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$ is Weihrauch equivalent to $\Pi^1_1\text{-}\mathsf{Tot}_2$ and $\Pi^1_1\text{-}\mathsf{DNC}_2$. Moreover, we have also shown in Proposition 2.9 that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$ is Weihrauch equivalent to $\Pi^1_1\text{-}\mathsf{Tot}_{\mathbb{N}}$. Recall from Section 2.3 we have introduced the Π^1_1-diagonalization principle $\Pi^1_1\text{-}\mathsf{DNC}_{\mathbb{N}}$, which is a special case of the cofinite Σ^1_1-choice principle. For this principle, we know in advance a bound of the number of elements removed by a cofinite set. Such a principle is bounded by the following principle for $\ell \in \mathbb{N}$:

$$\Sigma^1_1\text{-}\mathsf{C}_X^{\mathsf{cof}\restriction\ell} = \Sigma^1_1\text{-}\mathsf{C}_X\restriction_{\{A\subseteq X:|X\setminus A|\leq\ell\}}.$$

Namely, we have $\Pi^1_1\text{-}\mathsf{DNC}_{\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}^{\mathsf{cof}\restriction 1}$. One can consider the coproduct of $(\Sigma^1_1\text{-}\mathsf{C}_X^{\mathsf{cof}\restriction\ell})_{\ell\in\mathbb{N}}$ and call it *strongly-cofinite choice* on X. One can show that this principle is strictly weaker than the cofinite choice (Proposition 3.8 and Theorem 3.17 below). Even more generally, we consider *finite-or-strongly-cofinite choice*, denoted $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fosc}}_{\mathbb{N}\to\mathbb{N}}$, which accepts an input of the form (p,ψ), where for any $n\in\mathbb{N}$, $p(n)$ is a code of a Σ^1_1 subset $S_{p(n)}$ of $\mathbb{N}$ such that either $S_{p(n)}$ is nonempty and finite, or $|\mathbb{N}\setminus S_{p(n)}|\leq\psi(n)$. If (p,ψ) is an acceptable input, then $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fosc}}_{\mathbb{N}\to\mathbb{N}}$ chooses one element from $\prod_n S_{p(n)}$.

We show that all-or-unique choice is already strong enough to compute finite-or-strongly-cofinite choice:

Proposition 3.8. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fosc}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$.

Proof. First, it is clear that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fosc}}_{\mathbb{N}\to\mathbb{N}} \geq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ as witnessed by the indentity function. It remains to prove $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fosc}}_{\mathbb{N}\to\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$. Let $A=\prod_n A_n$ with a bound ψ given. We will construct a uniformly Σ^1_1 sequence $(B^n_m)_{m\leq\psi(n)}$ of subsets of $\mathbb{N}$. We use $B^n_0, B^n_1,\dots,B^n_{\psi(n)-1}$ to code information which element is removed from A_n whenever A_n is cofinite, and use $B^n_{\psi(n)}$ to code full information of A_n whenever A_n is finite. If a_0 is the first element removed from A_n, then put $B^n_0=\{a_0\}$, and if a_1 is the second element removed from A_n, then put $B^n_1=\{a_1\}$, and so on. If A_n becomes a finite set, then $B^n_{\psi(n)}$ just copies A_n, otherwise $B^n_{\psi(n)}=\mathbb{N}$. One can easily ensure that for any $n\in\mathbb{N}$ and $m<\psi(n)$, if A_n is finite, then B^n_m is a singleton disjoint from A_n; otherwise $B^n_m=\mathbb{N}$. Moreover, we can also see that either $B^n_{\psi(n)}$ is nonempty and finite or $B^n_{\psi(n)}=\mathbb{N}$.

Now, assume that $X\in\prod_{n,m}B^n_m$ is given. If $X(n,\psi(n))\notin\{X(n,i):i<\psi(n)\}$, then put $Y(n)=X(n,\psi(n))$. Otherwise, choose $Y(n)\notin\{X(n,i):i<\psi(n)\}$. Clearly, the construction of Y from X is uniformly computable.

If A_n becomes a finite set, the first case happens, and $Y(n)=X(n,\psi(n))\in B^n_{\psi(n)}=A_n$. If A_n remains cofinite, it is easy to see that $\mathbb{N}\setminus A_n\subseteq\{X(n,i):i<\psi(n)\}$, and therefore $Y(n)\in A_n$. Consequently, $Y\in A$. □

Corollary 3.9. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fosc}}_{\mathbb{N}\to\mathbb{N}}$.

In the following we will only consider all-or-finite choice, by convenience. We now prove that dependent choice does not add any power, and the existence of an instance that codes all the other, with very similar proofs as in the $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}}$ case.

Theorem 3.10. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{aof}}_{\mathbb{N}}$.

Proof. It is clear that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{aof}}_{\mathbb{N}}$. The argument for $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{aof}}_{\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ is similar to the finite case (Theorem 3.4). If T is a Σ^1_1 tree, for every σ define T_σ by the following Σ^1_1 procedure: First, put $T_\sigma=\mathbb{N}$, and then wait for $\mathrm{succ}_T(\sigma):=\{n:\sigma^\frown n\in T\}$ to be finite but nonempty. As in the proof of Theorem 3.4, if $\mathrm{succ}_T(\sigma)$ is finite, its finiteness is witnessed at some stage $<\omega_1^{\mathrm{CK}}$ by Spector bounding (Theorem 1.4). For the first such stage α (which is not necessarily a successor ordinal), if $\mathrm{succ}_T(\sigma)$ becomes empty, we just keep $T_\sigma=\mathbb{N}$, and end the construction. If $\mathrm{succ}_T(\sigma)$ turns out to be finite but nonempty, at every stage after α define T_σ to be $\mathrm{succ}_T(\sigma)$ except if this one becomes empty. Note that if $\mathrm{succ}_T(\sigma)$ becomes a finite set at some stage α_0, but an empty set at a later stage α_1, then the least such stage α_1 must be a successor ordinal, and therefore we can keep T_σ being nonempty (see also the proof of Theorem 3.4). Clearly, T_σ is either finite or $\mathbb{N}$. If T is an instance of $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{aof}}_{\mathbb{N}}$, then $T_\sigma = \mathrm{succ}_T(\sigma)$ and therefore $\prod_{\sigma\in\mathbb{N}^{<\mathbb{N}}} T_\sigma \geq_M [T]$. □

The absence of a maximal element in the Medvedev semi-lattice of the axiom of choice on "all-or-finite" sets would allow us to Weihrauch separate it from its "finite" version. However, $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ does also have a maximum element.

Theorem 3.11. *There exists a single maximum Medvedev degree in* $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ *and* $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{aof}}_{\mathbb{N}}$.

Proof. The argument is again similar to Theorem 3.4, even though we have no compactness assumption.

By Theorem 3.10 that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \equiv_W \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{aof}}_{\mathbb{N}}$, it suffices to prove the result for one of the functions, let us say $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$. Let $A_e = \prod_n S^e_n$ be the e-th Σ^1_1 homogeneous set. We set $\tilde{A}_e = \prod_n \tilde{S}^e_n$ to be defined by the following Σ^1_1 procedure: First, set $\tilde{S}^e_n = \mathbb{N}$, and then wait for some S^e_n to become finite and nonempty. If this happens, define $\tilde{S}^e_n = S^e_n$ until it removes its last element. At this point, leave $\tilde{S}^e_n$ nonempty, which is possible since it can happen only at a successor stage (this is because S^e_n has already become a finite set at some previous stage). Then, one can see that $\tilde{A}_e$ is in $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$, and if A_e is also an instance of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ then $\tilde{A}_e = A_e$.

Then $(\tilde{A}_e)_{e\in\mathbb{N}}$ is an enumeration of all nonempty elements of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$. Define the maximum to simply be $\prod_e \prod_n \tilde{S}^e_n$. □

We now prove that the relaxed constraint on the sets that allows them to be full does increase the power of the choice principle, making $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ strictly above $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$. We use the fact that the semi-lattice of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$ has a maximal element (Theorem 3.4), and we show that it must be strictly below some instance of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$.

Theorem 3.12. *For every* $A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$*, there exists* $B \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ *such that* $A <_M B$.

Proof. We will find $C = \prod_n C_n \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ such that $C \not\leq_M A$. Then, $B = A \times C$ will witness the theorem.

Now, let us describe the co-enumeration of C_n. First, wait for $x \mapsto \Phi^x_n(n)$ to be total on A, where Φ_n is the n-th partial computable function: More precisely, as $\Phi^x_n(n)\downarrow$ is Σ^0_1, the formula $\forall x(x \in A \to \Phi^x_n(n))$ is Π^1_1. Hence, if it holds, it is witnessed at some stage before ω_1^{CK} (via a Π^1_1 norm assigned to this formula). Next, since A is compact and Φ is continuous, Φ takes only finitely many values on A; that is, $\Phi^A_n(n) := \{\Phi^x_n(n) : x \in A\}$ is finite. Clearly, $\Phi^A_n(n)$ is Σ^1_1, and therefore, again, as in the proof of Theorem 3.4, the finiteness of $\Phi^A_n(n)$ is witnessed at some stage α before ω_1^{CK}. At this point, remove everything from C_n except $\max \Phi^A_{n,\alpha}(n) + 1$, where $\Phi^A_{n,\alpha}(n)$ is the stage α approximation of $\Phi^A_n(n)$.

We have that C_n is either $\mathbb{N}$ if the co-enumeration is stuck waiting for $x \mapsto \Phi^x_n(n)$ to be total on A, or a singleton otherwise. Also, it is clear that for any n, Φ_n cannot be a witness that $C \leq_M A$, so $C \not\leq_M A$. □

Corollary 3.13. *We have* $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}} <_W \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$.

Proof. By Theorem 3.12 and Theorem 3.4. □

One can also use the domination property to separate the all-or-finite choice principle and the (σ-)compact principle.

Proposition 3.14. *There exists* $A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ *such that every element* $p \in A$ *computes a function which dominates all* Δ^1_1 *functions.*

Proof. Let $(\varphi_e)_{e\in\mathbb{N}}$ be an effective enumeration of all partial Π^1_1 functions on $\mathbb{N}$. Define $A_{\langle e,k\rangle} \subseteq \mathbb{N}$ for $k, e \in \mathbb{N}$ as follows. Begin with $A_{\langle e,k\rangle} = \mathbb{N}$. Wait until we see $\varphi_e(k)\downarrow$. If it happens, set $A_{\langle e,k\rangle} = \{\varphi_e(k)\}$. Define $A = \prod_n A_n$. Then define $\Psi^p(k) = \sum_{e\le k} p(\langle e,k\rangle)$, which is clearly computable in p. It is easy to see that Ψ^p dominates all Δ^1_1 function whenever $p \in A$. □

This shows that all-or-finite Σ^1_1-choice is not Weihrauch-reducible to σ-compact Σ^1_1-choice.

Corollary 3.15. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \not\le_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Proof. Recall that a computable instance of $\Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is a countable union of compact Σ^1_1 sets. Thus, by Lemma 3.5, there is a solution p to any given computable instance of $\Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ such that any function which is higher Turing reducible to p is majorized by a Δ^1_1 function. However, by Proposition 3.14, there is a computable instance of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aou}}_{\mathbb{N}\to\mathbb{N}}$ whose solution consists of Δ^1_1 dominants. □

Corollary 3.16. $\Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *is not parallelizable, and* $\Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Proof. Clearly, $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ (and therefore $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$) is Weihrauch reducible to the parallelization of $\Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$. Therefore, by Corollary 3.15, $\Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is not parallelizable. By definition, any $\Sigma^1_1\text{-}\mathsf{AC}^{\triangledown}_{\mathbb{N}\to\mathbb{N}}$ is parallelizable, and so is $\Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$ by Proposition 2.5. □

3.3. **Axioms of cofinite analytic choice.** The choice problem when all sets are cofinite is quite different from the other restricted choices we study. It is the only one that does not include $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$.

Let us fix an instance $A = \prod_n A_n$ of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$. For every n, A_n is cofinite, so there exists a_n such that for any $i \ge a_n$, we have $i \in A_n$. Now, call f the function $n \mapsto a_n$. We have that $f \in A$, and for every g pointwise above f, we must have $g \in A$. So we clearly have $A \le_{\mathsf{W}} \{g \in \mathbb{N}^\mathbb{N} : \forall i\ f(i) \le g(i)\} = A_f$. This essential property of $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ prevents an instance to have more computational power than A_f for some $f \in \mathbb{N}^\mathbb{N}$.

The cofiniteness still allows some more power, as we will prove in this section that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ is Weihrauch incomparable with both $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$ and $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$.

Theorem 3.17. *There exists* $A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ *such that for any* $B \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ $A \not\le_M B$.

Proof. We use the existence of a maximum all-or-finite degree of Theorem 3.11 to actually only prove

$$\forall B \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}, \exists A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}} : A \not\le_M B.$$

Fix a $B = \prod_{n\in\mathbb{N}} B_n$, with $B_n \subseteq \mathbb{N}$ being either $\mathbb{N}$ or finite. We will construct $A = \prod_{e\in\mathbb{N}} A_e$, and use A_e to diagonalize against Φ_e being a witness for the reduction, by ensuring that either Φ_e is not total on B, or $\exists k \in \mathbb{N}, \sigma \in \prod_{n<k} B_n$ with $\Phi_e^\sigma(e) \downarrow \notin A_e$. Here is a description of the construction of A_e, along with sequences of string (σ_n) and (τ_n):

(1) First, $A_e = \mathbb{N}$. Wait for a stage where $B \subseteq \mathrm{dom}(\Phi_e)$, that is Φ_e is total on the current approximation of B. Define $\sigma_0 = \epsilon = \tau_0$, and move to step (2).
(2) Let n be the maximum such that τ_n is defined. Find $\sigma_{n+1} \succ \tau_n$ in the current approximation of B such that $\Phi_e^{\sigma_{n+1}}(e) \downarrow$. Take σ_{n+1} to be the leftmost such, and remove $\Phi_e^{\sigma_{n+1}}(e)$ from A_e. Move to step (3).
(3) Wait for some stage where $\Phi_e^B(e) := \{\Phi_e^X(e) : X \in B\} \subseteq A_e$. If it happens, wait again for the current approximation of B to be "all or finite", which will happen. Take τ_{n+1} to be the greatest prefix of σ_{n+1} still in B, and return to step (2).

As in the proof of Theorem 3.12, the property $B \subseteq \mathrm{dom}(\Phi_e)$ is Π^1_1, so it is witnessed at some stage $< \omega_1^{\mathrm{CK}}$. Let us prove that A_e is cofinite. If the co-enumeration of A_e stays at step (1), then $A_e = \mathbb{N}$ is cofinite. Otherwise, let us prove that there can only be finitely many τ_n defined.

Suppose all τ_n are defined. Then, the pointwise limit of the τ_n must be defined: Let ℓ be an integer such that $(\tau_n(\ell'))_{n\in\mathbb{N}}$ stabilizes for big enough n, for all $\ell' < \ell$. Start from a stage n_0 where they have stabilized. For $n > n_0$, if $\tau_{n+1}(\ell)$ changes, that is $\tau_{n+1}(\ell) \neq \tau_n(\ell)$, it must be that $\tau_n(\ell)$ has been removed from B_ℓ. But then, B_ℓ will become finite before the co-enumeration continues, and $\tau_n(\ell)$ can only take values in B_ℓ and never twice the same. Therefore, $(\tau_n(\ell))_n$ becomes constant at some point. Now, let $X \in \mathbb{N}^\mathbb{N}$ be the limit of (τ_n). As $X(\ell) \in B_\ell$ for every ℓ, we have $X \in B$. Since $B \subseteq \mathrm{dom}(\Phi_e)$ is already witnessed at some previous stage, and this is a positive property, we must have $X \in \mathrm{dom}(\Phi_e)$. Thus, there is $\sigma \prec X$ such that $\Phi_e^\sigma(e) \downarrow$. Let s be such that $\sigma \preceq \tau_t$ for any $t \geq s$. However, our algorithm can reach step (2) at most once after s: This is because, as $\sigma \preceq \tau_s \preceq \sigma_{s+1}$, we must have $\Phi_e^{\sigma_{s+1}}(e) = \Phi_e^\sigma(e)$, which is removed from A_e at step (2). This ensures that $\Phi_e^X(e) = \Phi_e^\sigma(e) \notin A_e$, so $\Phi_e^B(e) \subseteq A_e$ is never witnessed, and thus τ_{s+1} is undefined.

Hence, there is an n_0 such that τ_n is defined only for $n < n_0$, and thus σ_n can be defined only for $n \leq n_0$, therefore by construction at most $n_0 + 1$ elements are removed from A_e, and thus A_e is cofinite.

It remains to prove that $A \not\leq_M B$. Suppose Φ_e is a witness for the Medvedev reduction. Φ_e must be total on B, so we get past step (1) in the definition of A_e. Then, as only finitely many τ_n are defined, the co-enumeration has to be stuck at some step, waiting for something to happen. It cannot be stuck in step (2), as Φ_e is total on B, and any finite sequence $\tau \in \prod_{n<|\sigma|} B_n$ can be extended in an element of B by homogeneicity. This means that the co-enumeration is stuck at step (3), waiting for $\Phi_e^B(e) \subseteq A_e$, to never happen. This leaves us with $\Phi_e^B(e) \not\subseteq A_e$, so there is $X \in B$ such that $\Phi_e^X(e) \notin A_e$, and so $\Phi_e^X \notin A$. □

Theorem 3.18. *For any $A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ and $B \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$, if $A \leq_M B$, then A contains a Δ^1_1 path.*

Proof. Assume that $A \leq_M B$ via some partial computable function Φ, and A and B are of the forms $\prod_n A_n$ and $\prod_n B_n$, respectively. We describe the Δ^1_1 procedure to define $C \in A$:

Given n, in parallel, wait for n to be enumerated in one of those two Π^1_1 sets:

(1) If n is enumerated in $\{n : \exists k \in \mathbb{N}, \forall f \in \mathbb{N}^{\mathbb{N}}, \exists \sigma \geq f,\ \Phi^{\sigma}(n) = k\}$, define $C(n)$ to be one of these k.
(2) If n is enumerated in $\{n : \forall f \in \mathbb{N}^{\mathbb{N}}, \forall k, \exists k' > k, \exists \sigma \geq f$ such that $\Phi^{\sigma}(n) = k'\}$ then define $C(n) = 0$.

Here, $\sigma \geq f$ denotes the pointwise domination order, that is, $\sigma(n) \geq f(n)$ for all $n < |\sigma|$. As the items (1) and (2) are both Π^1_1, it is easy to check that C is Δ^1_1.

We claim that one of the two options will happen. Assume that case (2) fails. Then, there are $f \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$ such that for any $\sigma \geq f$, if $\Phi^{\sigma}(n)\downarrow$ then $\Phi^{\sigma}(n) \leq k$. If moreover (1) fails, then for any $\ell \leq k$, there is f_ℓ such that if $\sigma \geq f_\ell$ and $\Phi^{\sigma}(n)\downarrow$ then $\Phi^{\sigma}(n) \neq \ell$. We define $g \in \mathbb{N}^{\mathbb{N}}$ by $g(j) = \max\{f(j), f_0(j), \dots, f_k(j)\}$ for any j. Then we have $g \geq f, f_0, \dots, f_k$, and therefore, if $\sigma \geq g$ then $\Phi^{\sigma}(n)$ cannot take any value. Hence, as B_j is cofinite for any j, there is $b_j \in \mathbb{N}$ such that $[b_j, \infty) \subseteq B_j$. As Φ is total on B, if $h(j) \in B_j$ for every j then $\Phi^h(n)$ is defined for any n. However, if we define $h(j) = \max\{g(j), b_j\}$ then $h(j) \in B_j$ and therefore $\Phi^h(n)\downarrow$, a contradiction. This verifies our claim.

As before, by cofiniteness, there exists $b \in \mathbb{N}^{\mathbb{N}}$ such that $g \geq b$ implies $g \in B$. Fix n. In case (1), for $k = C(n)$, there is $\sigma \geq b$ such that $\Phi^{\sigma}(n) = k$. Therefore, $C(n) \in \Phi^B(n) := \{\Phi^X(n) : X \in B\}$, and moreover $\Phi^B(n) \subseteq A_n$ since Φ witnesses $A \leq_M B$ and $A = \prod_n A_n$. Hence we get $C(n) \in A_n$. In case (2), there are infinitely many k and there is $\sigma \geq b$ such that $\Phi^{\sigma}(n) = k$. This means that $\Phi^B(n)$ is infinite, and therefore A_n is infinite since $\Phi^B(n) \subseteq A_n$ as above. Hence $A_n = \mathbb{N}$ and $C(n) \in A_n$. Consequently, we obtain $C \in A$. □

Corollary 3.19. *We have both* $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}} \not\leq_W \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$ *and* $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}} \not\leq_W \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$.

Proof. The first part is implied by Theorem 3.17 (see the argument in the paragraph below Definition 3.1). The second part is implied by Theorem 3.18 and the fact that there exist Σ^1_1 finitely branching homogeneous trees with no Δ^1_1 member (cf. [14, Theorem 4.3 and Lemma 4.4]). □

We now show that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ does not admit a maximal element, using a proof similar to the one of Theorem 3.17.

Theorem 3.20. *For every* $B \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$*, there exists* $A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ *such that* $A \not\leq B$*. Thus,* $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ *does not have a maximal element.*

Proof. Fix a $B = \prod_{n\in\mathbb{N}} B_n$, with each $B_n \subseteq \mathbb{N}$ cofinite. We will construct $A = \prod_{e\in\mathbb{N}} A_e$, and use A_e to diagonalize against Φ_e being a witness for the reduction, by ensuring that either Φ_e is not total on B, or $\exists k \in \mathbb{N}, \sigma \in \prod_{n<k} B_n$ with $\Phi^{\sigma}_e(e)\downarrow \notin A_e$. Here is a description of the construction of A_e, along with sequences of string (σ_n) and (τ_n):

(1) First, $A_e = \mathbb{N}$. Wait for a stage where $B \subseteq \mathrm{dom}(\Phi_e)$, that is Φ_e is total on the the current approximation of B. Define $\sigma_0 = \epsilon = \tau_0$ and move to step (2).
(2) Let n be the maximum such that τ_n is defined. Find $\sigma_{n+1} \succ \tau_n$ in the current approximation of B such that $\Phi^{\sigma_{n+1}}_e(e)\downarrow$. Take σ_{n+1} to be the leftmost such, and remove $\Phi^{\sigma_{n+1}}_e(e)$ from A_e. Move to step (3).
(3) Wait for some stage where $\Phi^B_e(e) \subseteq A_e$. Take τ_{n+1} to be the greatest prefix of σ_{n+1} still in B, and return to step (2).

Let us prove that A_e is cofinite. If the co-enumeration of A_e stays at step (1), then $A_e = \mathbb{N}$ is cofinite. Otherwise, let us prove that there can only be finitely many τ_n defined, just as in Theorem 3.17.

Suppose infinitely many (τ_n) are defined. Then, this must have a limit: Let ℓ be a level such that $(\tau_n(\ell'))_n$ stabilizes for all $\ell' < \ell$. Start from a stage where they have stabilized. From this stage, if $\tau_n(\ell)$ changes, it must have been removed from B_ℓ. But that can happen only finitely many times, as B_ℓ is cofinite. Therefore, $(\tau_n(\ell))$ becomes constant at some point. However, this is impossible:Let $X \in \mathbb{N}^\mathbb{N}$ be the limit of (τ_n). As $X(\ell) \in B_\ell$ for every ℓ, we have $X \in B$. Since $B \subseteq \mathrm{dom}(\Phi_e)$ is already witnessed at some previous stage, and this is a positive property, we must have $X \in \mathrm{dom}(\Phi_e)$. Thus, there is $\sigma \prec X$ such that $\Phi_e^\sigma(e)\downarrow$. Let s be such that $\sigma \preceq \tau_t$ for any $t \geq s$. However, our algorithm can reach step (2) at most once after s: This is because, as $\sigma \preceq \tau_s \preceq \sigma_{s+1}$, we must have $\Phi_e^{\sigma_{s+1}}(e) = \Phi_e^\sigma(e)$, which is removed from A_e at step (2). This ensures that $\Phi_e^X(e) = \Phi_e^\sigma(e) \notin A_e$, so $\Phi_e^B(e) \subseteq A_e$ is never witnessed, and thus τ_{s+1} is undefined. Hence, there is an n_0 such that τ_n is defined only for $n < n_0$, and thus σ_n can be defined only for $n \leq n_0$, therefore by construction at most $n_0 + 1$ elements are removed from A_e, and thus A_e is cofinite.

Hence, there is an n_0 such that τ_n is defined only for $n < n_0$, and thus σ_n can be defined only for $n \leq n_0$, therefore by construction at most $n_0 + 1$ elements are removed from A_e, and thus A_e is cofinite.

It remains to prove that $A \not\leq_M B$. Suppose Φ_e is a potential witness for the inequality. Either Φ_e is not total on B, or we get stuck at some step in the co-enumeration of A_e, waiting for $\Phi_e^B(e) \subseteq A_e$ to never happen, leaving us with $\Phi_e^B(e) \not\subseteq A_e$, so there is $X \in B$ such that $\Phi_e^X(e) \notin A_e$, and so $\Phi_e^X \notin A$.

We now prove the last assertion: $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ does not have a maximal element. Let $B = \prod_n B_n \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$. Let $A = \prod_n A_n$ given by the first part of the proof. Then, $C = \prod_n A_n \times \prod_m B_m \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ and $A, B \leq_M C$ so $B <_M C$. $\square$

We here also note some domination property of the cofinite choice. The following fact is implicitly proved by Kihara-Marcone-Pauly [14, Lemma 4.7] to separate $\Sigma^1_1\text{-}\mathsf{WKL}$ and $\widehat{\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}}}$.

Fact 3.21 ([14])**.** There exists $A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ such that every element $p \in A$ computes a function which dominates all Δ^1_1 functions.

Therefore, as in the proof of Corollary 3.15, we can observe the following.

Corollary 3.22. $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}} \not\leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{K}_\sigma\mathsf{C}_{\mathbb{N}}$.

3.4. **Axioms of finite-or-cofinite analytic choice.** In this part, we study the weakened restriction to sets that are either finite, or cofinite. This restriction allows any instance from the stronger restrictions, thus $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}}$, $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$, and $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ are Weihrauch reducible to $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}}$ (and similarly for dependent choice). It is the weakest form of restriction other than "no restriction at all" that we will consider. However, we don't know if this restriction does remove some power and is strictly below $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ or not, as asked in Question 3.24.

In the following, we will show that $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}}$, $\Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{foc}}_{\mathbb{N}}$ and $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$, $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ do not admit a maximal computable instance. We will give several different proofs of this result. Theorem 3.23 is an attempt to answer Question 3.24 positively, but the conclusion turns out to be too weak, by lacking the effectivity required for a diagonalization.

Theorem 3.23. *For every $A \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}}$, there exists $B \in \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ such that $B \not\leq_M A$.*

Proof. We will build $B = \prod_e B_e \in \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ by defining B_e in a uniform Σ^1_1 way, such that if Φ_e is total on A, then $\Phi^A_e(e) \not\subseteq B_e$.

Fix $e \in \mathbb{N}$, and $A = \prod_n A_n \in \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}}$. In our definition of the co-enumeration of B_e along the ordinals, there will be two main steps: The first one forces that if $\Phi^A_e(e) \subseteq B_e$, then for every l, $|\Phi^{A\restriction_{\leq l}}_e(e)| < \omega$ where $A\restriction_{\leq l} = \{\sigma \in \mathbb{N}^{\leq l} : [\sigma] \cap A \neq \emptyset\}$. The second step will force that if $\Phi^A_e(e) \subseteq B_e$, then A is empty or Φ_e is not total on A.

In order to conduct all these steps, we will need to remove several times an element of B_e, but we do not want it to become empty. This is why in parallel of removing elements from B_e, we also mark some as "saved for later", so we know that even after infinitely many removals, B_e is still infinite.

We now describe the first part of the co-enumeration. For clarity, we use the formalism of an infinite time algorithm, that could easily be translated into a Σ^1_1 formula.

for $l \in \mathbb{N}$ **do**
  Mark a new element of B_e as saved;
  while $\Phi^{A\restriction_{\leq l}}_e(e)$ *is infinite* **do**
    for $i \in \mathbb{N}$ **do**
      Mark a new element of B_e as saved;
      Remove from B_e the first element of $\Phi^{A\restriction_{\leq l}}_e(e)$ that is not saved, if it exists. Otherwise, exit the loop;
      Wait for $\Phi^A_e(e) \subseteq B_e$;
    end
    Wait for every A_n with $n \leq l$ to be finite or cofinite;
    Unmark the elements marked as saved by the "for $i \in \mathbb{N}$" loop;
  end
end

Let us first argue that for a fixed l, the "while" part can only be executed a finite number of times. At every execution of the "for $i \in \mathbb{N}$" loop, either one element of $A\restriction_{\leq l}$ is removed, or $\Phi^{A\restriction_{\leq l}}_e(e)$ is finite and we exit the while loop (this is because at every step, only finitely many elements are marked as saved). But this means that if a "for" loop loops infinitely many times, by the pigeon hole principle there must exists a specific level $l_0 \leq l$ such that A_{l_0} went from cofinite to finite. But this can happen only $l + 1$ times, and the "while" loop can only run $l + 1$ many times.

Let us now argue that at every stage of the co-enumeration, including its end, every B_e is infinite. Fix a level l, and suppose that at the beginning of the corresponding "while" loop, B_e is infinite. As after every loops of the "for $i \in \mathbb{N}$" loop one element is saved, it means that after all these infinitely many loop, B_e contains infinitely many elements. This will happen during only finitely many loops of the "while" loop, so at the beginning of level $l + 1$, B_e is infinite. A similar argument with the elements saved by the first "for $l \in \mathbb{N}$" loop shows that if the first part of the co-enumeration ends, B_e is still infinite.

Now we split into two cases. If the first part of the co-enumeration never stops, as the "while" loop is in fact bounded, it means that the co-enumeration is forever stuck waiting for $\Phi_e^A(e) \subseteq B_e$. But as this never happens, B_e has the required property. Otherwise, the first part of the co-enumeration ends, and we are at a stage where for every l, $\Phi_e^{A\upharpoonright\leq l}(e)$ is finite, but B_e is infinite. We now continue to the second part of the co-enumeration of B_e:

for $l \in \mathbb{N}$ **do**

  Remove from B_e all the elements of $\Phi_e^{A\upharpoonright\leq l}(e)$;

  Wait for $\Phi_e^A(e) \subseteq B_e$;

end

We argue that this co-enumeration never finishes. Let $x \in A$, and $\sigma \prec x$ such that $\Phi_e^\sigma(e)\downarrow= k$. The co-enumeration will never reach the stage where $l = |\sigma + 1|$, as it cannot go through $l = |\sigma|$: If it reaches such stage, it will remove k from B_e and never have $\Phi_e^A(e) \subseteq B_e$. So, the co-enumeration has to stop at some step of the "for" loop, waiting for $\Phi_e^A(e) \subseteq B_e$ never happening. As B_e is infinite, it has the required property. □

In order to Weihrauch-separate $\Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}^{\mathsf{foc}}$ from the unrestricted $\Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$, one would need a stronger result with a single $B \in \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ not Medvedev reducible to any $A \in \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}^{\mathsf{foc}}$. We could try to apply the same argument to define $\prod_{\langle n,e\rangle} B_{\langle n,e\rangle}$, this time diagonalizing against an enumeration $(S^e)_{e\in\mathbb{N}}$ of $S^e = \prod_n S_n^e \in \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$. If S^e is not in $\Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}^{\mathsf{foc}}$, the co-enumeration will be stuck somewhere in the co-enumeration of some level, with no harm to the global diagonalization.

However, if some particular S^e is empty, we could end up with some $B_{\langle n,e\rangle} = \emptyset$, making B empty. Indeed, suppose we reach the second part of the co-enumeration. Then, the malicious S^e can make sure that every step of the second loop are achieved, by removing from S^e all strings σ such that $\Phi_e^\sigma(e)\downarrow\notin B_{\langle n,e\rangle}$, at every stage of the co-enumeration. As a result, both S^e and $B_{\langle n,e\rangle}$ will become empty.

Question 3.24. Do we have $\Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}^{\mathsf{foc}} <_{\mathsf{W}} \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$?

We now give a stronger result with a much simpler, but not effective, proof. As a corollary, we will obtain the fact that $\Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ and $\Sigma_1^1\text{-}\mathsf{DC}_{\mathbb{N}}$ do not admit a maximal computable instance.

Theorem 3.25. *For every $A \in \Sigma_1^1\text{-}\mathsf{DC}_{\mathbb{N}}$, there exists $B \in \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ such that $B \not\leq_M A$.*

Proof. We first claim that there is no enumeration of all nonempty elements of $\Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$. More than that, we will prove that there is no $\prod_{n,e\in\mathbb{N}} S_n^e \in \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ uniformly Σ_1^1 such that for every $B = \prod_n B_n \in \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$, there exists an e such that $\prod_n S_n^e \subseteq B$. Let $(S_n^e)_{n,e\in\mathbb{N}}$ be any uniformly Σ_1^1 enumeration. We construct $(B_e)_{e\in\mathbb{N}}$, a witness that this enumeration is not a counterexample to our claim. We define B_e by stages: At stage α, $B_e[\alpha]$ is equal to the open interval $]\min(S_e^e[\alpha]); \infty[$, where $S_e^e[\alpha]$ is the stage α approximation of S_e^e. Then $B_e = \bigcap_\alpha B_e[\alpha]$ defines a Σ_1^1 set. We have $\prod_n B_n \not\supseteq \prod_n S_n^e$ for every $e \in \mathbb{N}$ and the claim is proved.

Now, suppose that there exists $A \in \Sigma_1^1\text{-}\mathsf{DC}_{\mathbb{N}}$ such that for every $B \in \Sigma_1^1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$, we have $B \leq_M A$. Let us define S_n^e by

$$m \in S_n^e \iff \exists X \in A\ [\Phi_e^X(n)\downarrow= m \text{ or } \Phi_e \text{ is not total on } A].$$

Here, it is easy to check that non-totality of Φ_e on the Σ^1_1 set A is a Σ^1_1 property (see also the proof of Theorem 3.12). Hence, (S^e_n) is Σ^1_1 uniformly in e and n. Given any $B \in \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$, as $B \leq_M A$, fix a witness Φ_e. Then Φ_e is total on A, and therefore $S^e_n = \Phi^A_e(n) := \{\Phi^X_e(n) : X \in A\}$. By our choice of Φ_e, we also have $\Phi^A_e := \prod_n \Phi^A_e(n) \subseteq B$, and as B is homogeneous we obtain $\prod_n S^e_n \subseteq B$. Then, $(S^e_n)_{e,n\in\mathbb{N}}$ would be a contradiction to our first claim. □

Corollary 3.26. *Neither $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ nor $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ admits a maximal element.*

Proof. Let $A = \prod_n A_n \in \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$. By Theorem 3.25, let $B = \prod_n B_n \in \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ such that $B \not\leq_M A$. Then, $C = \prod_n A_n \times \prod_m B_m \in \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ is such that $A <_M C$. A similar argument works for $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$. □

There is another non-effective proof showing that $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ does not have a maximal element (but the proof does not work for $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$). Indeed, remarkably, the result shows that there is no greatest nonempty Σ^1_1 closed set even with respect to hyperarithmetical Muchnik degrees. We say that $A \subseteq \mathbb{N}^{\mathbb{N}}$ is *hyperarithmetically Muchnik reducible to* $B \subseteq \mathbb{N}^{\mathbb{N}}$ (written $A \leq^{\mathsf{HYP}}_w B$) if for any $x \in B$ there is $y \in A$ such that $y \leq_h x$, that is, y is hyperarithmetically reducible to x.

Fact 3.27 (cf. Gregoriades [11, Theorem 3.13])**.** If P is a Δ^1_1 closed set with no Δ^1_1 element, then there exists a clopen set C such that $P \cap C \neq \emptyset$ and $P <^{\mathsf{HYP}}_w P \cap C$.

Proof. This is what Gregoriades essentially obtained in the proof of [11, Theorem 3.13]. If $P = [T]$ for a computable tree T, the "key remark" in the proof of [11, Theorem 3.13] gives us a clopen neighborhood $C = [u]$ of some $\gamma \in P$ such that $\Delta^1_1(\gamma) \cap P \cap C = \emptyset$, so $\emptyset \neq P \cap C \not\leq^{\mathsf{HYP}}_w \{\gamma\} \subseteq P$. The tree T can be replaced with a Δ^1_1 tree since the "key remark" follows from the equivalence (1) in the proof of [11, Theorem 3.13], as it gives an implicit Σ^1_1 definition of the leftmost path α_L of P, which leads to a contradiction. Observe that, even if we replace T with a Δ^1_1 tree, the condition is still Σ^1_1, which concludes the proof. □

Note that any P satisfying the conclusion of the above fact cannot be homogeneous since if P is homogeneous, C is clopen, and $P \cap C$ is nonempty, then we always have $P \cap C \equiv_M P$. So, Fact 3.27 does not imply Theorem 3.25.

Corollary 3.28. *For any nonempty Σ^1_1 set $A \subseteq \mathbb{N}^{\mathbb{N}}$, there is a nonempty Π^0_1 set $B \subseteq \mathbb{N}^{\mathbb{N}}$ such that $A <^{\mathsf{HYP}}_w B$.*

Proof. For any nonempty Σ^1_1 set A, it is easy to see that there is a nonempty Π^0_1 set A^* such that $A \leq_M A^*$. If A^* has a Δ^1_1 element, then the assertion is clear as any Π^0_1 set B with no Δ^1_1 element is such that $A^* <^{\mathsf{HYP}}_w B$. If A^* has no Δ^1_1 element, by Fact 3.27, there is clopen C such that $A \leq_M A^* <^{\mathsf{HYP}}_w A^* \cap C$. □

In [5], Cenzer and Hinman showed that the lattice of Π^0_1 classes in Cantor space is dense. Here we already showed the lack of maximal elements, we now prove the lack of minimal elements:

Theorem 3.29. *For every $A \in \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ with no computable member, there exists $B >_M \mathbb{N}^{\mathbb{N}}$ in $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ such that*

$$\mathbb{N}^{\mathbb{N}} <_M A \cup B <_M A.$$

Proof. We first reduce the problem to finding a non-computable hyperarithmetical real X such that A contains no X-computable point. If such an X exists, then we have $\mathbb{N}^{\mathbb{N}} <_M A \cup \{X\} <_M A$.

It suffices to show that $\Phi_e^X \notin A$ for any e, and $\emptyset <_T X$. The latter condition is ensured by letting X be sufficiently generic. To describe a strategy for ensuring the first condition, fix a pruned Σ^1_1 tree T_A such that $[T_A] = A$. There are two ways for Φ_e to not be a witness that A has a X-computable element: either $\Phi_e^\sigma \notin T_A$ for some $\sigma \prec X$, or $X \notin \mathrm{dom}(\Phi_e)$. Let us argue that we have the following: For any $e \in \mathbb{N}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$ there exists a finite string τ extending σ such that

$$\text{either } \Phi_e^\tau \notin T_A \text{ or } [\tau] \cap \mathrm{dom}(\Phi_e) = \emptyset \tag{3}$$

Indeed, if it were not the case for some $e \in \mathbb{N}$, we would have a string σ such that for every τ extending σ, $\Phi_e^\tau \in T_A$ and there exists an extension $\rho \succ \tau$ such that Φ_e^ρ strictly extends Φ_e^τ, allowing us to compute a path of T_A, which is impossible as $A >_M \mathbb{N}^{\mathbb{N}}$.

Begin with the empty string $\sigma_0 = \emptyset$. For e let D_e be the e-th dense Σ^0_1 set of strings. Given σ_e, in a hyperarithmetical way, one can find a string $\sigma_e^* \in D_e$ extending σ_e. Now, we have a Π^1_1 function assigning e to the first σ_{e+1} extending σ_e^* we find verifying (3). This function is total, and then Δ^1_1. Moreover, it is clear that Φ_e^X does not define an element of A for any X extending σ_{e+1}. □

3.5. **Axiom of choice versus dependent choice.** H. Friedman showed that the axiom of Σ^1_1-dependent choice is strictly stronger than the axiom of Σ^1_1-choice in the context of second order arithmetic (cf. [19, Corollary VIII.5.14]). Although the Weihrauch degrees of the principles $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}^{\mathbb{N}}}$ and $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}^{\mathbb{N}}}$ are the same (Observation 2.1 and Proposition 2.2), we will see that $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ is strictly stronger than $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$, which finally solves Question 1.3:

Theorem 3.30. $\mathsf{ATR}_2 \not\leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$*; hence* $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$.

Proof. Let A_e be the set of solutions to the e-th computable instance of ATR_2, that is, $0^\frown H \in A_e$ if and only if H is a jump hierarchy for the e-th computable linear order $\prec_e$, and $1^\frown p \in A_e$ if p is an infinite decreasing sequence w.r.t. $\prec_e$. Suppose for the sake of contradiction that $A = \prod_e A_e$ is Medvedev reducible to a homogeneous Σ^1_1 set S. Let B be the set of all indices $e \in \mathbb{N}$ such that the set of all infinite decreasing sequences w.r.t. $\prec_e$ is not Medvedev reducible to S, and let C be the set of all indices $e \in \mathbb{N}$ such that the set of all jump-hierarchies for $\prec_e$ is not Medvedev reducible to S. Note that B and C are Σ^1_1.

Moreover, we claim that B and C are disjoint. To see this, let Φ be a continuous function witnessing $A \leq_M S$. If there is $X \in S$ such that $\Phi^X(0)$ is i then, by continuity of Φ, there is a finite initial segment σ of X such that $\Phi^Y(0) = i$ for any Y extending σ. However, by homogeneity of S, $S \cap [\sigma]$ is Medvedev equivalent to S. This means that, for any e, S Medvedev bounds either the set of infinite paths or the set of jump-hierarchies for the e-th computable tree. This concludes the proof of the claim.

Let WO be the set of all indices of well-orderings, and NPWO be the set of all indices for computable linear orderings with infinite hyperarithmetic decreasing sequences (i.e., linear orderings which are not pseudo-well-ordered). Clearly, WO is contained in B. Moreover, by H. Friedman's theorem [7] saying that a computable linear order which supports a jump hierarchy cannot have a hyperarithmetical descending sequence (see also Friedman [8] for a simpler proof based on Steel's result

[20]), NPWO is contained in C. Since B and C are disjoint Σ^1_1 sets, by the effective version of the Lusin separation theorem (cf. [17, Exercise 4B.11]), there is a Δ^1_1 set A separating B from C. This contradicts (Goh's refinement [10, Theorem 3.3] of) Harrington's unpublished result, which states that if a Σ^1_1 set separates WO from NPWO, then it must be Σ^1_1-complete. So A cannot be Medvedev below a homogeneous set, and thus $\mathsf{ATR}_2 \not\leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$.

Clearly, $\mathsf{ATR}_2 \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ since being a jump hierarchy and being an infinite decreasing sequence are arithmetical properties. Since $\Sigma^1_1\text{-}\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is Weihrauch equivalent to $\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$ by Proposition 2.2, we obtain $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}}$. $\square$

Finally, we decompose the axiom of countable Σ^1_1 choice into finite Σ^1_1 choice and cofinite Σ^1_1 choice.

Theorem 3.31. $\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}} \leq_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}} \star \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$.

Proof. Given a homogeneous Σ^1_1 tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, let f_T be the leftmost path through T. Then f_T has a finite-change higher approximation, i.e., there is a Δ^1_1 sequence approximating f with finite mind-changes (cf. [1] for the definition). Let $m_T(n)$ be the number of changes of the approximation procedure for $f_T \upharpoonright n+1$. One can assume that $f_T(n) \leq m_T(n)$. Then, one can effectively construct a Σ^1_1 sequence $(S_n)_{n\in\mathbb{N}}$ of cofinite subsets of $\mathbb{N}$ such that $m \in S_n$ implies $m > m_T(n)$. In particular, any element $g \in \prod_n S_n$ majorizes m_T, and thus f_T. Use $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}}$ to choose such a g, and consider the $\Sigma^1_1(g)$ tree $T^g = \{\sigma \in T : (\forall n < |\sigma|)\ \sigma(n) < g(n)\}$. Then T^g is a finite branching infinite tree since $f_T \in [T^g]$. Therefore, as in the proof of Proposition 2.5, one can effectively covert T^g into a $\Sigma^1_1(g)$ infinite binary tree T^*. Use $\Sigma^1_1\text{-}\mathsf{WKL}$ (which is Weihrauch equivalent to $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$, as seen in Theorem 3.3) to get an infinite path p through T^*. From p one can easily construct an infinite path through $T^g \subseteq T$. $\square$

3.6. **Summary of this section.** In summary, we obtain the following:

Theorem 3.32. *We have*

$$\begin{tikzcd}
\Sigma^1_1\text{-}\mathsf{DC}_{\mathbb{N}} \arrow[r] \arrow[d, Rightarrow] & \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{foc}}_{\mathbb{N}} \arrow[r, Rightarrow] \arrow[d] & \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{aof}}_{\mathbb{N}} \arrow[r, Rightarrow] \arrow[d, leftrightarrow] & \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{fin}}_{\mathbb{N}} \arrow[d, leftrightarrow] \\
\Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}} \arrow[r] & \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{aof}}_{\mathbb{N}\to\mathbb{N}} \arrow[r, Rightarrow] \arrow[d, "\times" description] & \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}} \\
 & & \Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}} \arrow[ur, "\times" description] &
\end{tikzcd}$$

Here, arrows $\to$ and $\Rightarrow$ denote $\geq_{\mathsf{W}}$ and $>_{\mathsf{W}}$, respectively. See also Figure 1.

A few questions about $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}}$ remain open:

Question 3.33. Is $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{foc}}_{\mathbb{N}}$? Is $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{foc}}_{\mathbb{N}\to\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$?

We also do not know if the dependent and independent choice for cofinite sets coincide.

Question 3.34. Is $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{cof}}_{\mathbb{N}\to\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{DC}^{\mathsf{cof}}_{\mathbb{N}}$?

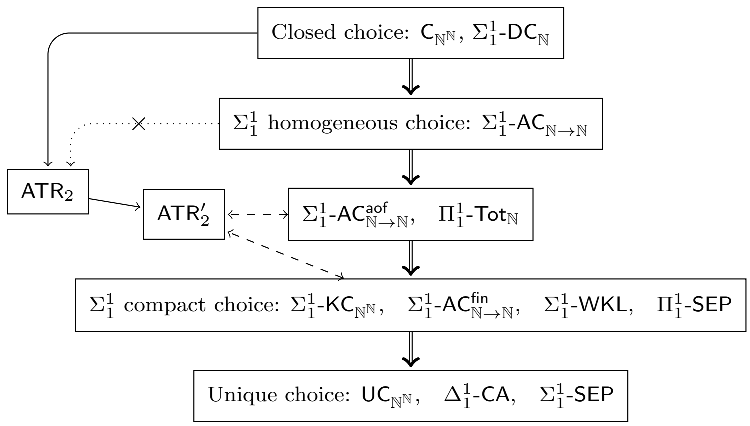

FIGURE 1. Key principles between $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (where dashed lines denote parallel arithmetical Weihrauch reducibility)

We solved the main question by showing Σ^1_1-$\mathsf{AC}_{\mathbb{N}\to\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1$-$\mathsf{DC}_\mathbb{N}$ (Theorem 3.30), but it is just a computable separation. Therefore, it is natural to ask if Σ^1_1-$\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ and Σ^1_1-$\mathsf{DC}_\mathbb{N}$ can be separated even in the hyperarithmetical sense. In other words, the following is one of the most important open questions, where $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is the unique choice principle (or equivalently, the choice principle for Σ^1_1 singletons; cf. [14]).

Question 3.35. Is $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \star \Sigma^1_1$-$\mathsf{AC}_{\mathbb{N}\to\mathbb{N}} <_{\mathsf{W}} \Sigma^1_1$-$\mathsf{DC}_\mathbb{N}$?

We also ask a question purely on the structure of Medvedev degrees for finite axioms of choice. Define more generally Σ^1_1-$\mathsf{AC}^{\mathsf{P}}_{\mathbb{N}\to\mathbb{N}}$ to be Σ^1_1-$\mathsf{AC}_{\mathbb{N}\to\mathbb{N}}$ where the set from which we choose have to be taken from P. For instance, if $\mathsf{P} = \{A \subseteq \mathbb{N} : |A| < \omega\}$, then Σ^1_1-$\mathsf{AC}^{\mathsf{P}}_{\mathbb{N}\to\mathbb{N}} = \Sigma^1_1$-$\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}}$.

Question 3.36. Let $\mathsf{P} = \{A \subseteq \mathbb{N} : A \subseteq 2\}$ and $\mathsf{Q} = \{A \subseteq \mathbb{N} : |A| \leq 2\}$. Is every element of Σ^1_1-$\mathsf{AC}^{\mathsf{Q}}_{\mathbb{N}\to\mathbb{N}}$ Medvedev equivalent to some element of Σ^1_1-$\mathsf{AC}^{\mathsf{P}}_{\mathbb{N}\to\mathbb{N}}$?

We are also interested in comparing various kinds of arithmetical transfinite recursion.

Question 3.37. $\mathsf{ATR}_2 \equiv^a_{\mathsf{W}} \mathsf{ATR}_{2'} \equiv^a_{\mathsf{W}} \mathsf{ATR}^{\mathsf{po}}_2$?

Finally, we mention a few descriptive set theoretic results deduced from our results. For a set $A \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$, each $A_x = \{y \in \mathbb{N}^\mathbb{N} : (x, y) \in A\}$ is called a *section*. If the section A_x is nonempty for each $x \in \mathbb{N}^\mathbb{N}$ then A is called *total*. Below, we use $\leq^{\mathsf{c}}_{\mathsf{W}}$ to denote the continuous version of Weihrauch reducibility; see [3].

Theorem 3.38. (1) *There is a total analytic set A with compact homogeneous sections such that any total analytic set with compact sections is $\leq^{\mathsf{c}}_{\mathsf{W}}$-reducible to A.*
(2) *For any total analytic set A with closed sections, there is a total analytic set with homogeneous sections which is not $\leq^{\mathsf{c}}_{\mathsf{W}}$-reducible to A.*
(3) *There is a total $F_{\sigma\delta}$ set with G_δ sections which is not $\equiv^{\mathsf{c}}_{\mathsf{W}}$-equivalent to any analytic set with closed sections.*
(4) *There is a total closed set which is not $\leq^{\mathsf{c}}_{\mathsf{W}}$-reducible to any total analytic set with homogeneous sections.*

Proof. (1) follows from the relativization of Theorem 3.4 since $\Sigma^1_1\text{-}\mathsf{AC}^{\mathsf{fin}}_{\mathbb{N}\to\mathbb{N}} \equiv_{\mathsf{W}} \Sigma^1_1\text{-}\mathsf{KC}_{\mathbb{N}^\mathbb{N}}$ by Proposition 2.5 and Theorem 3.3. (2) follows from the relativization of Theorem 3.25. For (3), let S be the set of pairs (x,y) with $y \not\leq_T x$. Then S is $F_{\sigma\delta}$, and each $S(x) = \{y : y \not\leq_T x\}$ is co-countable; hence G_δ. Suppose that S is $\equiv^c_{\mathsf{W}}$-equivalent to an analytic set A with closed sections. In particular, there are x-computable functions h_0, h_1 such that $S(x) \leq^x_M A(h_0(x)) \leq^x_M S(h_1 \circ h_0(x))$, where $\leq^x_M$ indicates the Medvedev reducibility relative to x. By definition of S, $z \leq_T x$ implies $S(z) \supseteq S(x)$, so $S(z) \leq_M S(x)$. Hence, $S(h_1 \circ h_0(x)) \leq_M S(x)$, and thus, we have $S(x) \equiv^x_M A(h_0(x))$. Since $A(h_0(x))$ is closed, one can apply (a relativization of) Theorem 3.29 to get a $\Sigma^1_1(x)$ closed set B such that $\mathbb{N}^\mathbb{N} <^x_M B <^x_M A(h_0(x))$. However, this implies $\mathbb{N}^\mathbb{N} <^x_M B <^x_M S(x)$, which is impossible by definition of S. Finally, (4) follows from the relativization of Theorem 3.30. □

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