A PRIORITY ARGUMENT IN DESCRIPTIVE SET THEORY (A VERY DETAILED EXPOSITION OF SEMMES' PROOF)

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Abstract. In 2009, Semmes announced that a function f on Baire space is decomposable into countably many Baire-one functions with G_δ domains if and only if the preimage of a F_σ set under f is $G_{\delta\sigma}$. In this report, I will outline Semmes' proof with the emphasis on the use of a finite injury priority argument, but not a game-theoretic one, and try to clarify how his argument works.

1. Introduction

1.1. **Background.** In 2009, Semmes [4] announced a result extending the Jayne-Rogers theorem [2]. Since then, a number of experts tried to clarify and simplify Semmes' proof, cf. [1]. Semmes' original exposition of his proof have laid emphasis on game-theoretic arguments. In this report, we will take a completely opposite approach: As is well known to experts, no determinacy argument has been used in Semmes' proof, and therefore, removing all the game-theoretic machineries makes the proof much clearer. Instead, we will put an emphasis on the use of *finite injury priority argument.*

In particular, we do not use Theorem 4.1.1 in Semmes [4] characterizing the Baire class 2 functions by the game $G_{1,3}$ (which is called Mistigri in [1]). This causes a few minor changes in the proof. For instance, our conditions $(*)$ and $(**)$ in pp. 9–10 are slightly different from the ones in Semmes [4, p.45 in Theorem 4.3.7]. As a consequence, the way of our exposition is slightly different from the original one; however, all of the essential ideas are already contained in Semmes' original insightful proof.

1.2. **Notations.** $[\sigma]$ is the clopen set generated by $\sigma \in \omega^{<\omega}$. An open set in ω^ω is said to be finitary if it is of the form $\bigcup_{\sigma\in F}[\sigma]$ for some finite set $F \subseteq \omega^{<\omega}$. If a string σ is an initial segment of τ then we write $\sigma \sqsubseteq \tau$. If strings σ and τ are incomparable then we write $\sigma\perp\tau$. For a function $f\colon X \to Y$ and $A \subseteq X$, we use $f|_A$ to denote the restriction of f up to A. Let $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ be pointclasses. We write $f^{-1}\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$ if the preimage of each $\mathbf{\Gamma}$ set under f is $\mathbf{\Lambda}$, that is,

$$A \in \mathbf{\Gamma} \implies f^{-1}[A] \in \mathbf{\Lambda} \text{ in } \mathrm{dom}(f).$$

For example, f is $\mathbf{\Sigma}^0_n$-measurable if and only if $f^{-1}\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_n$ holds.

If $\mathcal{F}$ is a class of functions, we also write $f \in \mathbf{dec}(\mathcal{F}/\mathbf{\Gamma})$ if f is decomposable into countably many $\mathcal{F}$-functions on $\mathbf{\Gamma}$ domains, that is, there is a countable $\mathbf{\Gamma}$ cover $(X_i)_{i\in\omega}$ of the domain of f such that $f|_{X_i} \in \mathcal{F}$ for each $i \in \omega$.

We also use $\mathbf{\Sigma}^0_n$ to denote the class of $\mathbf{\Sigma}^0_n$-measurable functions. For instance, the Jayne-Rogers theorem [2] can be stated as follows.

$$f^{-1}\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Sigma}^0_2 \iff \mathbf{dec}(\mathbf{\Sigma}^0_1/\mathbf{\Pi}^0_1)$$

where f is a function from an analytic subset of a Polish space to a separable metrizable space.

It is easy to see the following (see Motto Ros [3] and Semmes [4, Lemma 4.3.1])

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Observation 1. *The following equalities hold.*

$$\mathbf{dec}(\mathbf{\Sigma}^0_m/\mathbf{\Pi}^0_n)=\mathbf{dec}(\mathbf{\Sigma}^0_m/\mathbf{\Sigma}^0_{n+1})=\mathbf{dec}(\mathbf{dec}(\mathbf{\Sigma}^0_m/\mathbf{\Pi}^0_n)/\mathbf{\Sigma}^0_{n+1}).$$

2. Proof

In his PhD thesis [4], Semmes showed that the following equivalence holds.

$$f^{-1}\mathbf{\Sigma}^0_2\subseteq\mathbf{\Sigma}^0_3 \iff \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2).$$

The right-to-left implication is clear. Moreover, the condition $f^{-1}\mathbf{\Sigma}^0_2\subseteq\mathbf{\Sigma}^0_3$ always implies that f is $\mathbf{\Sigma}^0_3$-measurable. Thus, to verify the above equivalence, it suffices to show the following:

Theorem 2 (Semmes [4])**.** *Suppose that $f:\omega^\omega\to\omega^\omega$ is $\mathbf{\Sigma}^0_3$-measurable. Then,*

$$f\notin\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)\implies f^{-1}\mathbf{\Sigma}^0_2\not\subseteq\mathbf{\Sigma}^0_3.$$

2.1. **Transfinite derivation process.** To prove Theorem 2, hereafter we fix a $\mathbf{\Sigma}^0_3$-measurable function $f:\omega^\omega\to\omega^\omega$. Then, the preimage $f^{-1}[\sigma]$ of a clopen set is $\mathbf{\Sigma}^0_3$. Therefore, it can be written as a countable union of $\mathbf{\Pi}^0_2$ sets, say $f^{-1}[\sigma]=\bigcup_{s\in\omega}f^*_s[\sigma]$. This decomposition $f^*_s[\sigma]$ is the replacement for the game $G_{1,3}$ in [4] or the Mistigri in [1]. It looks too simple, but it certainly works.

Let $\mathbf{D}$ be a subset of ω^ω, and put $h=f|_{\mathbf{D}}$. Then, define $h^*_s[\sigma]=f^*_s[\sigma]\cap\mathbf{D}$. In Section 2.1, we will present an essence of the argument of Semmes [4, Lemma 4.3.3].

Given $X\subseteq\omega^\omega$, we define $[X;h]^\dagger_\sigma$ as follows:

$$[X;h]^\dagger_\sigma=X\setminus\bigcup\{J:h|_{\mathbf{D}\cap X\cap J\setminus h^{-1}[\sigma]}\in\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)\},$$

where J ranges over open sets in ω^ω. Moreover, given Y, we consider the following $[Y;h]^\star_{\sigma,s}$:

$$[Y;h]^\star_{\sigma,s}=\mathrm{cl}_Y(h^*_s[\sigma]),$$

where $\mathrm{cl}_Z A$ is the topological closure of a set $A\cap Z$ in a space Z. We call the above procedure a $\dagger_\sigma$*-derivation* (or a $\dagger$-derivation) and a $\star_{\sigma,s}$*-derivation* (or a $\star$-derivation), respectively. Clearly, $[X;h]^\dagger_\sigma$ and $[Y;h]^\star_{\sigma,s}$ are closed subsets of X and Y, respectively. In Semmes' thesis [4, Lemma 4.3.3], a $\dagger$-derivation and a $\star$-derivation are called a Ξ-operation and an Ω-operation, respectively.

We fix h, and simply write $X^\dagger_\sigma$ and $Y^\star_{\sigma,s}$ for $[X;h]^\dagger_\sigma$ and $[Y;h]^\star_{\sigma,s}$, respectively. We iterate these derivation procedures:

$$\begin{aligned}H^0_{\sigma,s}&=\omega^\omega,\\ H^{\alpha+1}_{\sigma,s}&=((H^\alpha_{\sigma,s})^\dagger_\sigma)^\star_{\sigma,s},\\ H^\alpha_{\sigma,s}&=\bigcap_{\beta<\alpha}H^\beta_{\sigma,s}\text{ if }\alpha\text{ is a limit ordinal.}\end{aligned}$$

Note that there is a countable ordinal $\gamma(\sigma,s)$ such that $H^{\gamma(\sigma,s)+1}_{\sigma,s}=H^{\gamma(\sigma,s)}_{\sigma,s}$ since $(H^\alpha_{\sigma,s})_\alpha$ is a decreasing sequence of closed sets in ω^ω. Clearly, $\gamma=\sup_{\sigma,s}\gamma(\sigma,s)+1$ is a countable ordinal since $\aleph_1$ is regular.

We divide the set $\mathbf{D}$ into three pieces. We first define the (σ,s)*-kernel* to be

$$K_{\sigma,s}h=H^{\gamma(\sigma,s)}_{\sigma,s}.$$

We say that a point $x\in X$ is *generic* if for every (σ,s), either $x\in h^{-1}[\sigma]$ or there exists α such that $x\in(H^\alpha_{\sigma,s})^\dagger_\sigma\setminus H^{\alpha+1}_{\sigma,s}$ (that is, x is removed by a $\star_{\sigma,s}$-derivation). Define $\mathbf{K}=\mathbf{D}\cap\bigcup_{\sigma,s}K_{\sigma,s}h$, $\mathbf{G}$ to be the set of all generic points $y\in\mathbf{D}\setminus\mathbf{K}$, and $\mathbf{A}$ to be the set of all other points in $\mathbf{D}$. Note that $x\in\mathbf{A}$ iff $x\notin h^{-1}[\sigma]$ holds, and x must be removed by a $\dagger$-derivation for some (σ,s).

Proposition 3.

(1) $\mathbf{A} \in \mathbf{\Delta}^0_3$, $\mathbf{G} \in \mathbf{\Delta}^0_3$, *and* $\mathbf{K} \in \mathbf{\Sigma}^0_2$, *in* $\mathbf{D}$.
(2) $h|_{\mathbf{G}}$ *is* $\mathbf{\Sigma}^0_2$*-measurable.*
(3) $h|_{\mathbf{A}} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$.

To see this, we need the following characterization of the set $\mathbf{A}$.

Claim. Suppose $x \in \mathbf{D} \setminus \mathbf{K}$. Then, $x \in \mathbf{A}$ if and only if there are $(\sigma_0, s_0), (\sigma_1, s_1)$ and $\alpha_0, \alpha_1 < \gamma$ such that $[\sigma_0] \cap [\sigma_1] = \emptyset$ and $x \in H^{\alpha_i}_{\sigma_i,s_i} \setminus (H^{\alpha_i}_{\sigma_i,s_i})^\dagger_{\sigma_i}$ for every $i \in \{0, 1\}$.

Proof. The condition $x \notin \mathbf{K}$ means that for every (σ, s), there is $\alpha(\sigma, s)$ such that $x \in H^{\alpha(\sigma,s)}_{\sigma,s} \setminus H^{\alpha(\sigma,s)+1}_{\sigma,s}$. Now $x \in \mathbf{D}$ and thus $h(x)$ is defined.

If $x \in \mathbf{A}$ then $x \notin h^{-1}[\sigma_0]$ holds, and x must be removed by a $\dagger$-derivation, that is, $x \in H^{\alpha_0}_{\sigma_0,s_0} \setminus (H^{\alpha_0}_{\sigma_0,s_0})^\dagger_{\sigma_0}$ where $\alpha_0 = \alpha(\sigma_0, s_0)$. Since $h(x) \notin [\sigma_0]$ and since σ_0 is nonempty, there is σ_1 such that $[\sigma_0] \cap [\sigma_1] = \emptyset$ and $h(x) \in [\sigma_1]$. Let s_1 be such that $x \in h^*_{s_1}[\sigma_1]$. Note that for any Z, $x \in Z$ clearly implies $x \in \mathrm{cl}_Z(h^*_{s_1}[\sigma_1])$. Therefore, x is not removed by a $\star_{\sigma_1,s_1}$-derivation, and thus x must be removed by a $\dagger$-derivation. More precisely, for all α, $x \in (H^{\alpha}_{\sigma_1,s_1})^\dagger_{\sigma_1}$ implies $x \in H^{\alpha+1}_{\sigma_1,s_1}$; hence by putting $\alpha_1 = \alpha(\sigma_1, s_1)$, we get $x \in H^{\alpha_1}_{\sigma_1,s_1} \setminus (H^{\alpha_1}_{\sigma_1,s_1})^\dagger_{\sigma_1}$ as desired.

We next verify the converse direction. Let x be a point in $\mathbf{D} \setminus \mathbf{K}$ satisfying the latter condition. Since $[\sigma_0] \cap [\sigma_1] = \emptyset$, we must have $x \notin h^{-1}[\sigma_i]$ for some $i < 2$. Then, the pair (σ_i, s_i) witnesses that x is removed by a $\dagger_{\sigma_i}$-derivation. This implies that x is not generic. Hence, under our assumption that $x \notin \mathbf{K}$, we have $x \in \mathbf{A}$ as desired. □

Proof of Proposition 3. (1) By definition, clearly $\mathbf{K} \in \mathbf{\Sigma}^0_2$ in $\mathbf{D}$. Hence, by the above claim, $\mathbf{A}$ is $\mathbf{\Sigma}^0_2$ in $\mathbf{D} \setminus \mathbf{K}$, and thus $\mathbf{A}$ is the difference of two $\mathbf{\Sigma}^0_2$ sets in $\mathbf{D}$. Then, $\mathbf{G}$ is also contained in a finite level of the difference hierarchy over $\mathbf{\Sigma}^0_2$ in $\mathbf{D}$.

(2) Suppose that $x \in \mathbf{G}$. Then, x is generic, and $x \notin \mathbf{K}$. Given (σ, s), Let $\alpha(\sigma, s)$ witness $x \notin \mathbf{K}$ as in the previous claim. If $x \in h^{-1}[\sigma]$, there is s such that $x \in h^*_s[\sigma]$ by definition. Then, as mentioned in the previous claim, x is not removed by a $\star$-derivation, and thus removed by a $\dagger$-derivation: For all α, $x \in (H^{\alpha}_{\sigma,s})^\dagger_{\sigma}$ implies $x \in H^{\alpha+1}_{\sigma,s}$; hence $x \in H^{\alpha(\sigma,s)}_{\sigma,s} \setminus (H^{\alpha(\sigma,s)}_{\sigma,s})^\dagger_{\sigma}$. If $x \notin h^{-1}[\sigma]$, by our definition of genericity, x is always removed by a $\star$-derivation: For all s, there exists α such that $x \in (H^{\alpha}_{\sigma,s})^\dagger_{\sigma} \setminus H^{\alpha+1}_{\sigma,s}$, which means that $x \in (H^{\alpha(\sigma,s)}_{\sigma,s})^\dagger_{\sigma} \setminus H^{\alpha(\sigma,s)+1}_{\sigma,s}$. Consequently, whenever $x \in \mathbf{G}$, for any σ,

$$x \in h^{-1}[\sigma] \iff (\exists \alpha < \gamma)(\exists s \in \omega)\ x \in H^{\alpha}_{\sigma,s} \setminus (H^{\alpha}_{\sigma,s})^\dagger_{\sigma}.$$

The latter condition is clearly $\mathbf{\Sigma}^0_2$.

(3) Let $(\sigma_i, s_i, \alpha_i)_{i<2}$ be a witness of $x \in \mathbf{A}$ as in the previous claim. By our definition of the $\dagger$-derivation, for $Z = H^{\alpha_0}_{\sigma_0,s_0} \cap H^{\alpha_1}_{\sigma_1,s_1}$, there is a neighborhood J of x such that

$$h|_{\mathbf{D} \cap Z \cap J \setminus h^{-1}[\sigma_i]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi^0_2}).$$

Clearly, $[\sigma_0] \cap [\sigma_1] = \emptyset$ implies $(J \setminus h^{-1}[\sigma_0]) \cup (J \setminus h^{-1}[\sigma_1]) = J$. Since $\mathbf{\Sigma}^0_3$-measurability of h and zero-dimensionality of ω^ω implies that $h^{-1}[\sigma_i]$ is $\mathbf{\Delta}^0_3$ in $\mathbf{D}$, both $J \setminus h^{-1}[\sigma_0]$ and $J \setminus h^{-1}[\sigma_1]$ are $\mathbf{\Delta}^0_3$ in $\mathbf{D}$. Hence, $h|_{\mathbf{D} \cap Z \cap J} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$. Since there are only countably many candidates for such a witness $(\sigma_i, s_i, \alpha_i)_{i<2}$ (because $\alpha_i < \gamma$), $h|_{\mathbf{A}}$ is decomposable into countably many $\mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$-functions on closed domains $(H^{\alpha_0}_{\sigma_0,s_0} \cap H^{\alpha_1}_{\sigma_1,s_1} \cap J)_{\sigma_0,s_0,\alpha_0,\sigma_1,s_1,\alpha_1,J}$. Hence, by Observation 1, we conclude $h|_{\mathbf{A}} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Delta^0_3})$. □

2.2. **Chain of kernels.** As a consequence of Proposition 3, we obtain the following key lemma.

Lemma 4. *If* $h \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, *then* $\mathbf{K}$ *is nonempty.*

Proof. If $\mathbf{K}$ is empty, then $h = h|_{\mathbf{A}} \cup h|_{\mathbf{G}}$. By Proposition 3 (2) and (3), we have $h|_{\mathbf{A}}, h|_{\mathbf{G}} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$. Since $\mathbf{A}$ and $\mathbf{G}$ are $\mathbf{\Delta}^0_3$ by Proposition 3 (1), we also have $h \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ by Observation 1. □

In particular, there is (σ, s) such that the (σ, s)-kernel $K_{\sigma,s}h$ has an intersection with $\mathbf{D}$. Note that $K_{\sigma,s}h$ is closed in ω^ω even if $\mathbf{D}$ is not. It is easy to see that $K_{\sigma,s}h \subseteq \mathrm{cl}_{\omega^\omega}\mathbf{D}$. This $K_{\sigma,s}$ corresponds to T in the statement of Semmes [4, Lemma 4.3.3]. Hereafter, if L is a closed set and p is a finite string, we write $p \in L$ if $L \cap [p] \neq \emptyset$, that is, we often identify a closed set with a pruned tree.

2.2.1. *The $\star$-derivation.* The $\star$-derivation procedure for $h = f|_{\mathbf{D}}$ ensures a *density* condition for $K_{\sigma,s}X$. This observation corresponds to the second property of T in Semmes [4, Lemma 4.3.3]. We say that a triple (K, σ, s) of a nonempty closed set K, a finite string σ, and a natural number s is a *bi-density triple* (w.r.t. f) if

(D1) $f^*_s[\sigma]$ is dense in K.
(D2) $f^*_t[\tau]$ is nowhere dense in K whenever $\sigma \perp \tau$ and $t \in \omega$.

Let $\mathbb{Q}$ be the set of all bi-density triples. The following proof is an analog of [4, Lemma 4.3.2].

Observation 5. *For any $\mathbf{D} \subseteq \omega^\omega$, σ and s, we have $(K_{\sigma,s}(f|_{\mathbf{D}}), \sigma, s) \in \mathbb{Q}$.*

Proof. Put $h = f|_{\mathbf{D}}$. The $\star$-derivation procedure clearly ensures that $h^*_s[\sigma]$ is dense in $K_{\sigma,s}h$. Hence $f^*_s[\sigma]$ is also dense in $K_{\sigma,s}h$, that is, (D1) holds. Suppose for the sake of contradiction that the item (D2) fails. Then $f^*_t[\tau]$ is dense in $K_{\sigma,s}h \cap [\eta]$ for some $\eta \in \omega^{<\omega}$, and $f^*_s[\sigma]$ is also dense in $K_{\sigma,s}h \cap [\eta]$ by (D1). By definition, $f^*[\sigma]$ and $f^*[\tau]$ are $\mathbf{\Pi}^0_2$ in the Polish space ω^ω. Thus, both are intersections of sequences of dense open sets in the closed set $K_{\sigma,s}h \cap [\eta]$. By the Baire category theorem, $f^*_s[\sigma]$ and $f^*_t[\tau]$ have an intersection. However, $\sigma \perp \tau$ implies $[\sigma] \cap [\tau] = \emptyset$ and thus we must have $f^*_s[\sigma] \cap f^*_t[\tau] \subseteq f^{-1}[\sigma] \cap f^{-1}[\tau] = \emptyset$. □

2.2.2. *The †-derivation.* The †-derivation procedure ensures an *indecomposability* condition for $K_{\sigma,s}h$. This observation corresponds to the first property of T in Semmes [4, Lemma 4.3.3]. We say that a triple $(L, p; V)$ of a nonempty closed set L, a finitary clopen set V, and a finite string $p \in L$ is an *indecomposability domain* (for f) if

$$(\forall q \in L)\ [q \sqsupseteq p \implies f|_{L \cap [q] \setminus f^{-1}[V]} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)].$$

Let $\mathbb{L}_1$ be the set of all indecomposability domains. The †-derivation procedure ensures the following.

Observation 6. *Assume that $f|_{\mathbf{D}} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, and that $\mathbf{D}$ and $f^{-1}[U]$ have no intersection. Then, there exists (σ, s) such that $(K_{\sigma,s}(f|_{\mathbf{D}}), \varepsilon; U \cup [\sigma]) \in \mathbb{L}_1$, where ε denotes the empty string.*

Proof. Put $h = f|_{\mathbf{D}}$. By Lemma 4, there is (σ, s) such that $K_{\sigma,s}h$ is nonempty. By definition of a kernel $K = K_{\sigma,s}h$, we have $K^\dagger_\sigma = K$. Note that $\mathbf{D} \cap f^{-1}[U] = \emptyset$ implies that $\mathbf{D} \setminus f^{-1}[U \cup \sigma] = \mathbf{D} \setminus f^{-1}[\sigma] = \mathbf{D} \setminus h^{-1}[\sigma]$, and f and h agrees on this set. Therefore, by definition of the †-derivation procedure, we have the following.

$$f|_{K \cap [q] \setminus f^{-1}[U \cup \sigma]} \supseteq f|_{\mathbf{D} \cap K \cap [q] \setminus f^{-1}[U \cup \sigma]} = h|_{\mathbf{D} \cap K \cap [q] \setminus h^{-1}[\sigma]} \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$$

for any $q \in \omega^{<\omega}$. This means that $(K_{\sigma,s}h, \varepsilon; U \cup [\sigma])$ is an indecomposability domain. □

The following states the basic properties of $\mathbb{L}_1$. The proof of the latter corresponds to [4, Lemma 4.3.1].

Observation 7. (1) *If $(L, p; V) \in \mathbb{L}_1$, then for any $V' \subseteq V$ and $q \in L$ with $q \sqsupseteq p$, we have $(L, q; V') \in \mathbb{L}_1$.*

(2) *For any pair* (J_0, J_1) *of disjoint finitary clopen sets, if* $(L, p; V) \in \mathbb{L}_1$ *then there are* $q \in L$ *with* $q \sqsupseteq p$ *and* $i < 2$ *such that* $(L, q; V \cup J_i) \in \mathbb{L}_1$.

Proof. (1) Obviously, if $f|_A \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ and $A \subseteq B$ then $f|_B \notin \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$. Now, note that $L \cap [q] \setminus f^{-1}[V] \subseteq L \cap [p] \setminus f^{-1}[V']$.

(2) Otherwise, $(L, p; V \cup J_0) \notin \mathbb{L}_1$ means that $f|_{L\cap[q]\setminus f^{-1}[V\cup J_0]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ for some $q \in L$ with $q \sqsupseteq p$, and similarly, $(L, q; V \cup J_1) \notin \mathbb{L}_1$ means that $f|_{L\cap[r]\setminus f^{-1}[V\cup J_1]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$ for some $r \in L$ with $r \sqsupseteq q$. Since J_0 and J_1 are disjoint, we have

$$(L \cap [r]) \setminus f^{-1}[V] = (L \cap [r] \setminus f^{-1}[V \cup J_0]) \cup (L \cap [r] \setminus f^{-1}[V \cup J_1]).$$

By $\mathbf{\Sigma}^0_3$-measurability of f, $f^{-1}[V \cap J_i]$ is $\mathbf{\Delta}^0_3$, and therefore, again by Observation 1, we can see that $f|_{L\cap[r]\setminus f^{-1}[V]} \in \mathbf{dec}(\mathbf{\Sigma}^0_2/\mathbf{\Pi}^0_2)$, and thus $(L, p; V) \notin \mathbb{L}_1$ since $p \sqsubseteq r \in L$. □

Definition 8. We say that $((L_\ell)_{\ell\leq a}, p^a; V)$ is an indecomposability layer if

$$\begin{gathered} L_0 \supseteq L_1 \supseteq \cdots \supseteq L_{a-1} \supseteq L_a, \\ (L_a, p^a; V) \text{ is an indecomposability domain, i.e., in } \mathbb{L}_1, \\ (\forall q^a \sqsupseteq p^a)(\exists r^{a-1} \sqsupseteq q^a)\ ((L_\ell)_{\ell<a}, r^{a-1}; V) \text{ is an indecomposability layer,} \end{gathered}$$

where q^a ranges over L_a and r^{a-1} ranges over L_{a-1}. Let $\mathbb{L}$ be the set of all indecomposability layers.

According to Semmes' terminology, $((L_\ell)_{\ell\leq a}, p^a; V)$ is an indecomposability layer iff p^a is $(L_\ell)_{\ell\leq a}$-V^{c}-good. The following corresponds to Semmes [4, Proposition 4.3.4].

Observation 9. *If* $((L_\ell)_{\ell\leq a}, p^a; V) \in \mathbb{L}$, $q^a \in L_a$, *and* $q^a \sqsupseteq p^a$, *then* $((L_\ell)_{\ell\leq a}, q^a; V) \in \mathbb{L}$

Proof. By Observation 7 (1). □

The following is a restatement of Semmes [4, Lemma 4.3.6] in our language, which generalizes Observation 7 (2).

Lemma 10. *Let* $(J_k)_{k<m}$ *be a collection of pairwise disjoint finitary clopen sets. If* $(\mathcal{L}, p^a; V) \in \mathbb{L}$, *then for all but* $|\mathcal{L}|$ *many indices* $i < m$, *we have* $(\mathcal{L}, q^a; V \cup J_i) \in \mathbb{L}$ *for some* $q^a \sqsupseteq p^a$.

Proof. First assume that $|\mathcal{L}| = 1$. In this case, $(\mathcal{L}, p; V) \in \mathbb{L}$ just means that $(\mathcal{L}, p; V) \in \mathbb{L}_1$. Thus, Observation 7 (2) clearly implies the assertion for $|\mathcal{L}| = 1$.

Consider the length $|\mathcal{L}| = a + 1$. In this proof, superscripts of variables x^a, y^{a-1}, etc. will indicate that x^a ranges over L_a, y^{a-1} ranges over L_{a-1}, etc. We put $S^i_a = \{q^a : (L_a, q^a; V \cup J_i) \in \mathbb{L}_1\}$ and $S^i_{<a} = \{q^{a-1} : ((L_\ell)_{\ell<a}, q^{a-1}; V \cup J_i) \in \mathbb{L}\}$. By Observations 7 (1) and 9, S^i_a and $S^i_{<a}$ are open in L_a. By induction, we assume the assertion for the length a, and fix $((L_\ell)_{\ell\leq a}, p^a; V) \in \mathbb{L}$. Since the third condition of $\mathbb{L}$ implies that, given $q^a \sqsupseteq p^a$, $((L_\ell)_{\ell<a}, r^{a-1}; V) \in \mathbb{L}$ for some $r^{a-1} \sqsupseteq q^a$, we get the following by the induction hypothesis:

(IH) Given $q^a \sqsupseteq p^a$, there is $r^{a-1} \sqsupseteq q^a$ such that for all but a many indices i, $u^{a-1} \in S^i_{<a}$ for some $u^{a-1} \sqsupseteq r^{a-1}$, that is, $S^i_{<a} \cap [r^{a-1}] \neq \emptyset$.

Note that (IH) implies that given $q^a \sqsupseteq p^a$, for all but a many indices i, $S^i_{<a} \cap [q^a] \neq \emptyset$.

Now we start to verify the assertion. By definition of $\mathbb{L}$, it suffices to show the following for all but $a + 1$ many indices $i < m$:

$$(\exists q^a \sqsupseteq p^a)\ [q^a \in S^i_a \text{ and } (\forall r^a_0 \sqsupseteq q^a)(\exists r^{a-1}_1 \sqsupseteq r^a_0)\ r^{a-1}_1 \in S^i_{<a}]. \tag{1}$$

Case 1. For any $i \leq m$, S^i_a is dense in $L_a \cap [p^a]$.

In this case, $\bigcap_{i<m} S^i_a$ is also dense in $L_a \cap [p^a]$ since the intersection of finitely many dense open sets is again dense. Let E be the set of all indices j such that the condition (1) fails. If $j \in E$, then since S^j_a is dense in $L_a \cap [p^a]$, the failure of (1) implies that

$$(\forall q^a_0 \sqsupseteq p^a)(\exists r^a_0 \sqsupseteq q^a_0)(\forall r^{a-1}_1 \sqsupseteq r^a_0)\ r^{a-1}_1 \notin S^j_{<a}.$$

Consider $\mathrm{ext}_{a-1} S^j_{<a} = \{q^{a-1} \in L_{a-1} : (\forall r^{a-1} \sqsupseteq q^{a-1})\ r^{a-1} \notin S^j_{<a}\}$, the exterior of $S^j_{<a}$ in L_{a-1}. Clearly, $\mathrm{ext}_{a-1} S^j_{<a}$ is open, and the above formula says that if $j \in E$, then $\mathrm{ext}_{a-1} S^j_{<a}$ is dense in $L_a \cap [p^a]$. Therefore, $\bigcap_{j\in E} \mathrm{ext}_{a-1} S^j_{<a}$ is also dense in $L_a \cap [p^a]$. In particular, there is $q^a_* \sqsupseteq p^a$ such that

$$(\forall j \in E)\ S^j_{<a} \cap [q^a_*] = \emptyset.$$

However, by (IH), for all but a many indices j, $S^j_{<a} \cap [q^a_*] \neq \emptyset$. Therefore, we have $|E| \leq a$.

Case 2. There is $i < m$ such that S^i_a is not dense in $L_a \cap [p^a]$.

In this case, $L_a \setminus S^i_a$ contains a nonempty open subset of $[p^a]$ in L_a, that is, there is $q^a \sqsupseteq p^a$ such that $q^a_0 \in L_a \setminus S^i_a$ for all $q^a_0 \sqsupseteq q^a$. By Observation 7 (2), if $j \neq i$, then for any $q^a_0 \sqsupseteq q^a$, there is $q^a_1 \sqsupseteq q^a_0$ such that $q^a_1 \in S^j_a$. In particular, $S^j_a \subseteq L_a$ is dense in $L_a \cap [q^a]$. Hence, $\bigcap_{i\neq j<m} S^j_a$ is dense in $L_a \cap [q^a]$.

Let E be the set of all indices j such that the condition (1) fails. If $j \in E$, $j \neq i$, then since S^j_a is dense in $L_a \cap [q^a]$, the failure of (1) implies that

$$(\forall q^a_0 \sqsupseteq q^a)(\exists r^a_0 \sqsupseteq q^a_0)(\forall r^{a-1}_1 \sqsupseteq r^a_0)\ r^{a-1}_1 \notin S^j_{<a}.$$

Thus, by the similar argument as before, we get $q^a_* \sqsupseteq p^a$ such that

$$(\forall j \in E \setminus \{i\})\ S^j_{<a} \cap [q^a_*] = \emptyset.$$

As before, (IH) implies $|E \setminus \{i\}| \leq a$. Hence, $|E| \leq a + 1$. This concludes the proof. □

2.2.3. *Semmes conditions.* We now introduce a key notion, which we call a Semmes condition.

Definition 11. A tuple $((L_\ell, \sigma_\ell, s_\ell)_{\ell\leq a}, p^a, V)$ is called a *Semmes condition* if

$$\begin{gathered} ((L_\ell)_{\ell\leq a}, p^a, V) \text{ is an indecomposability layer, i.e., in } \mathbb{L}, \\ (\sigma_\ell)_{\ell\leq a} \text{ is pairwise incomparable, and } \sigma_\ell \in V, \\ (L_\ell, \sigma_\ell, s_\ell) \text{ is a bi-density witness, i.e., in } \mathbb{Q}, \text{ for all } \ell \leq a. \end{gathered}$$

Let $\mathbb{S}$ be the set of all Semmes conditions. We say that $((T_\ell, \sigma_\ell, s_\ell)_{\ell\leq a}, q^a, V') \in \mathbb{S}$ *extends* $((T_\ell, \sigma_\ell, s_\ell)_{\ell<a}; p^{a-1}, V) \in \mathbb{S}$ if

$$T_a \subseteq T_{a-1},\ V' \supsetneq V,\ \sigma_a \notin V, \text{ and } q^a \sqsupseteq p^{a-1}$$

We will need to ensure that a Semmes condition always has an extension. We utilize the indecomposability condition to construct an extension. The following lemma is buried in Semmes [4, p.37 in Theorem 4.3.7]

Lemma 12. *Let $\mathcal{T}$ and $\mathcal{L}_i$, $i < c$, be bi-density witnesses, and $(\mathcal{T}, p^{a-1}, V), (\mathcal{L}_i, p_i, V)$ are Semmes conditions. Then, there are $\mathcal{T}' \supseteq \mathcal{T}$, $q^a \sqsupseteq p^{a-1}$, $p'_i \sqsupseteq p_i$, and $V' \supseteq V$ such that $(\mathcal{T}', q^a, V')$ extends $(\mathcal{T}, p^{a-1}, V)$, and $(\mathcal{L}_i, p'_i, V')$ are still Semmes conditions.*

Proof. Let $\mathcal{T} = (\mathcal{T}_\ell)_{\ell<a}$ be given, where $\mathcal{T}_\ell$ is of the form $(T_\ell, \tau_\ell, t_\ell)$. We will construct T_a. We claim that for any z, there is a sequence $(T^{n+1}_a, q^a_{n+1}, \sigma_n)_{n\leq z}$ such that $(q^a_{n+1})_{n\leq z}$ is increasing,

$$\begin{gathered} T_{a-1} \supseteq T^0_a \supseteq T^1_a \supseteq \cdots \supseteq T^z_a \supseteq T^{z+1}_a, \\ (\sigma_n)_{n\leq z} \text{ is pairwise incomparable,} \\ (T^{n+1}_a, q^a_{n+1}; V \cup [\sigma_n]) \text{ is an idecomposable domain, i.e., in } \mathbb{L}_1, \text{ for any } n \leq z, \end{gathered}$$

We first define $T_a^0 = T_{a-1}$, $q_0^a = p^{a-1}$, and $U_0 = \emptyset$. Since $(\mathcal{T}, p^{a-1}, V) \in \mathbb{S}$, $(T_a^0, q_0^a; V \cup U_0)$ is an indecomposability domain. Assume that we have constructed $(T_a^{s+1}, q_{s+1}^a, \sigma_s)_{s<n}$ fulfilling the above claim. Put $U_n = \bigcup_{s<n}[\sigma_s]$. Inductively assume that $(T_a^n, q_n^a; V \cup U_n)$ is an indecomposability domain, that is, $f|_{\mathbf{D}} \notin \mathbf{dec}(\mathbf{\Sigma}_2^0/\mathbf{\Pi}_2^0)$, where $\mathbf{D} = T_a^n \cap [q_n^a] \setminus f^{-1}[V \cup U_n]$. By applying Lemma 4 to $h = f|_{\mathbf{D}}$, one obtains σ_n, s_n such that

$$\emptyset \neq T_a^{n+1} := K_{\sigma_n, s_n} h \subseteq \mathrm{cl}_{\omega^\omega}(\mathbf{D}) \subseteq T_a^n \cap [q_n^a].$$

Since $\mathbf{D}$ and $f^{-1}[V \cup U_n]$ have no intersection, by Observation 6, $(T_a^{n+1}, \varepsilon; V \cup U_{n+1})$ is an indecomposability domain, and so is $(T_a^{n+1}, q_n^a; V \cup [\sigma_n])$ by Observation 7 (1). This verifies the third requirement of the claim whenever $q_n^a \sqsubseteq q_{n+1}^a \in T_a^{n+1}$.

To ensure the second requirement of the claim, we may need to choose a subsequence of $(T_a^{s+1}, q_{s+1}^a, \sigma_s)_s$. As seen in the proof of Observation 5, $h_{s_n}^*[\sigma_n]$ is dense in T_a^{n+1}. In particular, $h^{-1}[\sigma_n] \supseteq h_{s_n}^*[\sigma_n]$ is nonempty. Since $h^{-1}[V \cup U_n]$ is empty, we have $[\sigma_n] \not\subseteq [V \cup U_n]$. Note that $V \cup U_n$ is generated by a finite set I_n of finite strings. Moreover, the condition $[\sigma_n] \not\subseteq [V \cup U_n]$ means that either σ_n is incomparable with any elements in I_n or σ_n is an initial segment of an element in I_n. However, since I_n is finite, there are finitely many σ_n satisfying the latter condition. Let $j(n)$ be 1 plus the number of such strings. Hence, given s, if t is sufficiently large, $t \geq s + j(s)$ say, then we must have σ_t is incomparable with any strings in L_s. In particular, σ_t is incomparable with σ_u for any $u < s$. Define $h(s) = \sum_{u \leq s} j(u)$. We now replace $(T_a^{n+1}, q_{n+1}^a, \sigma_n)$ with $(T_a^{j(n)+1}, q_{j(n)+1}^a, \sigma_{j(n)})_{n \leq z}$, which satisfies all conditions of the claim.

We now have two cases.

$$(\forall r^a \sqsupseteq q_n^a)(\exists u^{a-1} \sqsupseteq r^a)\ ((T_\ell)_{\ell<a}, u^{a-1}, V \cup [\sigma_n]) \in \mathbb{L},$$

where r^a ranges over T_a^{n+1}, and u^{a-1} ranges over T_{a-1}. In this case, by combining with the third condition of the previous claim, we get that $((T_\ell)_{\ell<a}{}^\frown T_a^{n+1}, q_n^a, V \cup [\sigma_n]) \in \mathbb{L}$. Then define $q_{n+1}^a = q_n^a$. Otherwise, we have

$$(\exists r^a \sqsupseteq q_n^a)(\forall u^{a-1} \sqsupseteq r^a)\ ((T_\ell)_{\ell<a}, u^{a-1}, V \cup [\sigma_n]) \notin \mathbb{L}.$$

In this case, we define $q_{n+1}^a = r^a$. In any case, $q_n^a \sqsubseteq q_{n+1}^a \in T_{n+1}^a$.

Let E be the set of all indices $n \leq z$ such that the second case applies. Recall that $((T_\ell)_{\ell<a}, q_{z+1}^a, V) \in \mathbb{L}$ and $(\sigma_n)_{n \leq z}$ is pairwise incomparable. Therefore, by Lemma 10, for all but a many indices $i \leq z$, we have $((T_\ell)_{\ell<a}, u^{a-1}, V \cup [\sigma_i]) \in \mathbb{L}$ for some $u^{a-1} \sqsupseteq q_{z+1}^a$. This means that $|E| \leq a$. Hence, for all but a many indices $n \leq z$, the first case applies, and we get that $((T_\ell)_{\ell<a}{}^\frown T_a^{n+1}, q_{n+1}^a, V \cup [\sigma_n]) \in \mathbb{L}$.

Put $z = a + b + 1$, where $b = \sum_{i<c} |\mathcal{L}_i|$. Then, $|z \setminus E| > b$. By Lemma 10, there is $n \in z \setminus E$ such that for all $i < c$, we have $(\mathcal{L}_i, p_i', V \cup [\sigma_n]) \in \mathbb{S}$ for some $p_i' \sqsupseteq p_i$. Finally, put $T_a = T_a^{n+1}$, $q_a = q_{n+1}^a$ and $V' = V \cup [\sigma_n]$. By our choice of (σ_n, s_n) and by Observation 5, we have $\mathcal{T}_a := (T_a, \sigma_n, s_n) \in \mathbb{Q}$. Put $\mathcal{T}' = (\mathcal{T}_\ell)_{\ell \leq a}$. Then, we get $(\mathcal{T}', q^a, V') \in \mathbb{S}$, and it extends $(\mathcal{T}, p^{a-1}, V)$ as desired. □

Later our construction will make an *injury* (in the sense of a priority argument), which may decreases the length of the Semmes condition. We say that $(\mathcal{T}', q^\ell, V') \in \mathbb{S}$ is a *shortening of* $(\mathcal{T}, p^a, V) \in \mathbb{S}$ if $\mathcal{T}'$ is an initial segment of $\mathcal{T}$, $V' = V$, and $q^\ell \sqsupseteq p^a$. The following corresponds to Semmes [4, Proposition 4.3.5].

Observation 13. *Every Semmes condition $(\mathcal{T}, p^a, V)$ has a shortening of length ℓ for any $\ell \leq |\mathcal{T}|$.*

Proof. Let a be the length of $\mathcal{T}$. Since $((T_j)_{j<a}, p^a, V) \in \mathbb{L}$, one can find a sequence $p^{a-1} \sqsubseteq p^{a-2} \sqsubseteq \ldots \sqsubseteq p^\ell$ such that $((T_j)_{j<a}, p^\ell, V) \in \mathbb{L}$. □

2.3. **Priority argument.** We are now ready to prove Theorem 2. Given a $\mathbf{\Sigma}^0_3$ set $U \subseteq \omega^\omega$ we will construct a continuous function $\psi : \omega^\omega \to \omega^\omega$ and a set $V \subseteq \omega^{<\omega}$ of strings such that

$$x \in U \iff \psi(x) \in f^{-1}[V]$$

for every $x \in \omega^\omega$, where $[V]$ is the open set generated by V. Thus, this will ensure that $f^{-1}[V]$ is $\mathbf{\Sigma}^0_3$-complete for some set V of strings, which implies $f^{-1}\mathbf{\Sigma}^0_2 \not\subseteq \mathbf{\Sigma}^0_3$. We first describe $\mathbf{\Sigma}^0_3$ sets $f^{-1}[V]$ and U as follows:

$$\begin{aligned} x \in U &\iff (\exists a)(\forall b)(\exists c)\ S(x,a,b,c), \\ y \in f^{-1}[V] &\iff (\exists \sigma \in V)(\exists i)(\forall j)(\exists k)\ Q(y,\sigma,i,j,k). \end{aligned}$$

Here, $y \in f^*_i[\sigma]$ iff for all j, there exists k such that $Q(y,\sigma,i,j,k)$, where S and Q are Δ_1 formulas (or equivalently, clopen sets). For this reason, one can assume monotonicity of Q, that is, if $Q(y,\sigma,i,j,k)$ and $\tau \sqsubseteq \sigma$ then $Q(y,\tau,i,j,k)$ also holds, since replacing $Q(y,\sigma,i,j,k)$ with the condition $\exists \tau \sqsubseteq \sigma\ Q(y,\tau,i,j,k)$ does not affect the above property.

Requirements. The a-th requirements for our construction are given as follows:

$$\begin{aligned} \mathcal{N}^x_a : &\quad (\forall a' < a)(\exists b)(\forall c)\ \neg S(x,a,b,c) \implies (\forall \sigma \in V_{s_a})(\forall i < a)(\exists j)(\forall k)\ \neg Q(\psi(x),\sigma,i,j,k), \\ \mathcal{P}^x_a : &\quad (\forall b)(\exists c)\ S(x,a,b,c) \implies (\exists \sigma_a)(\exists i_a)(\forall j)(\exists k)\ Q(\psi(x),\sigma_a,i_a,j,k), \end{aligned}$$

where s_a is the first stage at which an a-th strategy acts along x (after the last initialization which may be caused by a higher-priority strategy; the details will be explained later).

Roughly speaking, every a-th strategy believes that a is the least witness for $x \in U$. Then, the $\mathcal{P}$-action tries to keep $\psi(x) \in f^*_{i_a}[\sigma_a]$ and the $\mathcal{N}$-action forces $\psi(x) \notin \bigcup_{i<a} f^*_i[V_{s_a}]$. These requirements ensure that such ψ is a desired reduction as follows.

- If a is the smallest witness for $x \in U$, then the requirement $\mathcal{P}^x_a$ ensures that $\psi(x) \in f^*_{i_a}[\sigma_a]$. The a-th strategy will put σ_a into the set V in the construction, so by the requirement $\mathcal{P}^x_a$, we get that $x \in U$ implies $\psi(x) \in f^{-1}[V]$.
- If $x \notin U$, then for every a, the premise of the requirement $\mathcal{N}_a$ must be true, and then, the combination of requirements $\mathcal{N}^x_a$'s will eventually ensure that $\psi(x) \notin f^{-1}[V] = \bigcup_a f^{-1}[V_{s_a}]$.

Thus, it suffices to describe the strategy to satisfy the requirements $\mathcal{P}^x_a$ and $\mathcal{N}^x_a$. A rough idea is to assign a Semmes condition $(\mathcal{T}_\alpha, p_\alpha, V_\alpha)$ to each string $\alpha \sqsubset x$, and the a-th tree T_a in the layer follows the a-th strategy.

Conditions. Fix a bijection $h : \omega^{<\omega} \to \omega$ such that $\alpha \sqsubseteq \beta$ implies $h(\alpha) \leq h(\beta)$. We simply write $\alpha \leq \beta$ if $h(\alpha) \leq h(\beta)$. At stage s, we will deal with the s-th binary string w.r.t. this order $\leq$. Given α, we use symbols $\alpha - 1$ and α^- to denote the immediate $\leq$-predecessor and the immediate $\sqsubseteq$-predecessor, respectively. At the α-th stage, we will construct $\mathcal{T}_\alpha$, $(p^\alpha_\beta)_{\beta \leq \alpha}$, and V_α satisfying the following condition.

- $(\mathcal{T}_\beta, p^\alpha_\beta, V_\alpha)$ is a Semmes condition for every $\alpha \in \omega^{<\omega}$ and $\beta \leq \alpha$.
- If $\beta \leq \alpha$, then $p^{\alpha-1}_\beta \sqsubseteq p^\alpha_\beta$ and $V_{\alpha-1} \subseteq V_\alpha$.
- $(\mathcal{T}_\alpha, p^\alpha_\alpha, V_\alpha)$ is either an extension or a shortening of $(\mathcal{T}_{\alpha^-}, p^{\alpha-1}_{\alpha^-}, V_{\alpha-1})$.

Then, we will define $V = \bigcup_{\alpha \in \omega^{<\omega}} V_\alpha$, and $\psi(x) = \bigcup_{s \in \omega} p^{x \upharpoonright s}_{x \upharpoonright s}$.

By the definition of a Semmes condition, $\mathcal{T}_\alpha$ is a sequence of bi-dense triples $(T_a, \sigma_a, i_a)_{a<\ell}$. Here, recall that (σ_a, i_a) witnesses the bi-dense property of T_a for every $a < \ell$:

$$\text{(D1)} \qquad (\forall q^a)(\exists y^a \sqsupset q^a)[\forall j \exists k Q(y^a, \sigma_a, i_a, j, k)],$$

$$\text{(D2)} \qquad (\forall \tau \perp \sigma_a)(\forall i)(\forall p^a)(\exists q^a \sqsupseteq p^a)(\forall y^a \sqsupset q^a)[\exists j \forall k \neg Q(y^a, \tau, i, j, k)],$$

where $p^a, q^a \in \omega^{<\omega}$ and $y^a \in \omega^\omega$ range over T_a.

We now start to describe the proof of Theorem 2. For the reader who is familiar with priority arguments in computability theory, we first note that our proof is a *finite injury priority argument.*

Proof of Theorem 2. As mentioned before, we will construct Semmes conditions $(\mathcal{T}_\alpha, p^\alpha_\beta, V_\alpha)_{\beta\leq\alpha}$ at the α-th stage. If ℓ is the length of $\mathcal{T}_{\alpha^-}$, then every strategy $a \leq \ell$ is *eligible to act* at the α-th stage, that is, we deal with $\mathcal{P}_a$- and $\mathcal{N}_a$-strategies for any $a \leq \ell$. The state of the a-th strategy at the α-th stage s is written as $\mathtt{state}(a, \alpha)$. If $a < \ell$ then $\mathtt{state}(a, \alpha)$ takes a value in ω, and if $a = \ell$, then $\mathtt{state}(a, \alpha) = \mathtt{init} \notin \omega$.

At the first stage ε, where ε is the empty string, we first use Lemma 4 to get σ, s such that $K_{\sigma,s}f$ is nonempty. Put $\mathcal{T}_\varepsilon = (K_{\sigma,s}f, \sigma, s)$, $p^\varepsilon_\varepsilon = \varepsilon$, and $V_\varepsilon = \{\sigma\}$, and then $(\mathcal{T}_\varepsilon, p^\varepsilon_\varepsilon, V_\varepsilon)$ forms a Semmes condition by Observations 5 and 6. We then set $\mathtt{state}(0, \varepsilon) = 0$ and $\mathtt{state}(a, \varepsilon) = \mathtt{init}$ for every $a > 0$.

At the beginning of stage α we inductively assume that $\mathtt{state}(a, \beta)$ has already been defined for any $\beta < \alpha$. By our assumption, a Semmes condition $\mathbf{p} = (\mathcal{T}_{\alpha^-}, p^{\alpha-1}_{\alpha^-}, V_{\alpha-1})$ has also been constructed by the previous stage. Let ℓ be the length of $\mathcal{T}_{\alpha^-}$, that is, $\mathcal{T}_{\alpha^-}$ is of the form $(\mathcal{T}_a)_{a<\ell}$ such that $\mathcal{T}_a = (T_a, \sigma_a, i_a) \in \mathbb{Q}$. At this stage α, we will consider $(\ell + 1)$ strategies.

A brief description of our strategies at stage α: Before giving the formal definition, we will explain an informal idea of our finite injury priority construction.

For $a < \ell$, the a-th strategy believes that a is the least witness for $x \in U$ under the current approximation $\alpha \sqsubseteq x$. Of course, the belief of the a-th strategy can be both correct for some $x \sqsupset \alpha$ and incorrect for some other $x' \sqsupset \alpha$, but she has the same belief at the current α. The a-th strategy looks for witnesses supporting her belief, and if she finds a new witness at the current stage α, she wants to act to fulfill the requirement $\mathcal{P}^x_a$ using the density (D1) of the a-th triple (T_a, σ_a, i_a).

The outmost strategy, i.e., the ℓ-th strategy, also believes that ℓ is the least witness for $x \in U$, but currently we do not have the ℓ-th level $(T_\ell, \sigma_\ell, i_\ell)$. The hope of the ℓ-th strategy at this stage is to construct a triple $(T_\ell, \sigma_\ell, i_\ell)$, and to make action under the belief that ℓ is the *least* witness, that is, no $a < \ell$ is a witness for $x \in U$. Then, the ℓ-th strategy needs to make sure the requirement $\mathcal{N}^x_\ell$ by using the nowhere density (D2) of a newly constructed $(T_\ell, \sigma_\ell, i_\ell)$.

Now, many strategies may want to act; however, their beliefs conflict with each other. Hence, we cannot allow more than one strategies to make action at one stage. To avoid such a conflict we put a priority order on strategies; a smaller $a \leq \ell$ has higher priority than a greater $a' \leq \ell$. Only the highest priority strategy among those who want to act can actually act. Thus, exactly one of the strategies acts at each stage. Along $x \in \omega^\omega$, if $x \in U$, then exactly one strategy has a correct belief, and we will see that such a strategy acts infinitely often. If $x \notin U$, then no one has a correct belief, and no strategy acts infinitely often.

We now give a formal description of the above argument. The stage α consists of $\ell + 1$ substages. At the a-th substage of stage α, we check whether the a-th strategy makes an action. We begin with substage $a = 0$, and consider the following actions.

Initial Action: If $\mathtt{state}(a, \alpha^-) = \mathtt{init}$, then $a = \ell$, so the ℓ-th strategy makes the following action.

(1) By Lemma 12, there are $\mathcal{T}_\ell = (T_\ell, \sigma_\ell, i_\ell)$, $p^\ell \sqsupseteq p^{\alpha-1}_\alpha$, $p^\alpha_\beta \sqsupseteq p^{\alpha-1}_\beta$, and $V_\alpha \supsetneq V_{\alpha-1}$ such that

$$((\mathcal{T}_a)_{a\leq\ell}, p^\ell, V_\alpha) \text{ extends } \mathbf{p} = (\mathcal{T}_{\alpha^-}, p^{\alpha-1}_{\alpha^-}, V_{\alpha-1}),$$
$$(\mathcal{T}_\beta, p^\alpha_\beta, V_\alpha) \text{ is a Semmes condition whenever } \beta < \alpha.$$

(2) To ensure $\mathcal{N}_\ell$, by the nowhere density condition (D2) for T_ℓ, since σ_ℓ is incomparable with any element in $V_{\alpha-1}$, there is a proper extension $q^\ell \sqsupset p^\ell$ in T_ℓ such that

$$(**) \qquad (\forall y^\ell \sqsupset q^\ell)(\forall \tau \in V_{\alpha-1})[\forall i < a \exists j \forall k \neg Q(y^\ell, \tau, i, j, k)],$$

where $y^\ell \in \omega^\omega$ ranges over T_ℓ. Note that $(**)$ means that $[q_\ell] \cap \bigcup_{i<a} f_i^*[V_{\alpha-1}] = \emptyset$, where one can think of $\tau \in V_{\alpha-1}$ as $[\tau] \subseteq [V_{\alpha-1}]$. Define $p^\alpha_\alpha = q^\ell$. By Observation 7 (1), $((\mathcal{T}_a)_{a\leq\ell}, p^\alpha_\alpha, V_\alpha)$ is still a Semmes condition, which extends the previous condition $\mathbf{p}$.

(3) Define $\mathtt{state}(\ell,\alpha) = 0$, and $\mathtt{state}(a,\alpha) = \mathtt{state}(a,\alpha^-)$ for $a \neq \ell$. Go to the next stage $\alpha + 1$.

The b-th Action: If $\mathtt{state}(a,\alpha^-) = b \in \omega$, then the a-strategy see if

$$(\forall b' \leq b)(\exists c \leq |\alpha|)\ S(\alpha, a, b', c).$$

If this does not hold, go to the next substage $a+1$. If this condition is true, the a-th strategy acts as follows. By Observation 13, we get a length $a+1$ shortening $((\mathcal{T}_b)_{b\leq a}, p^a, V_{\alpha-1}) \in \mathbb{S}$ of the previous condition $\mathbf{p}$. Now $\mathcal{T}_a$ is of the form $(T_a, \sigma_a, s_a) \in \mathbb{Q}$. By the density condition (D1) for T_a, there exists a proper extension $q^a \sqsupset p^a$ in T_a such that

$$(*) \qquad (\forall j \leq b)(\exists k)\ Q(q^a, \sigma_a, i_a, j, k).$$

Define $p^\alpha_\alpha = q^a$. By Observation 7, $((\mathcal{T}_b)_{b\leq a}, p^\alpha_\alpha, V_{\alpha-1})$ is a Semmes condition, which is a shortening of the previous condition $\mathbf{p}$. Then, define $\mathcal{T}_\alpha = (\mathcal{T}_b)_{b\leq a}$, $V_\alpha = V_{\alpha-1}$, $\mathtt{state}(a,\alpha) = b+1$, $\mathtt{state}(i,\alpha) = \mathtt{state}(i,\alpha^-)$ for any $i < a$ and $\mathtt{state}(i,\alpha) = \mathtt{init}$ for any $i > a$. Go to the next stage $\alpha+1$.

Outcomes: Put $V = \bigcup_{\alpha\in\omega^{<\omega}} V_\alpha$, and $\psi(x) = \bigcup_{s\in\omega} p^{x\upharpoonright s}_{x\upharpoonright s}$. Clearly, V is open since $(V_\alpha)_{\alpha\in\omega^{<\omega}}$ is a union of open sets. Moreover, ψ is continuous since $(p^{x\upharpoonright s}_{x\upharpoonright s})_{s\in\omega}$ is increasing.

Lemma 14. *For any x, we have the following.*

$$x \notin U \iff \lim_{s\to\infty} \mathtt{state}(a, x \upharpoonright s) \text{ converges for every } a.$$

Proof. ($\Rightarrow$) If $x \notin U$ then for any a there is b such that $(\exists c)S(x,a,b,c)$ fails. Then the b-th action never occur at any initial segment of x. Thus, we must have $\mathtt{state}(a, x\upharpoonright s) \leq b$ for any s. If the state of a strategy does not change, then it does not injure any other strategy. Therefore, by induction, we can see $\lim_s \mathtt{state}(a, x\upharpoonright s)$ converges for every a.

($\Leftarrow$) If $x \in U$ then there is a such that for every b, we have $(\exists c)S(x,a,b,c)$. Let a_0 be the smallest such a. Then it is easy to see that for any b, a_0 proceeds the b-th action at some stage s_b such that $\mathtt{state}(a_0, x\upharpoonright s_b) = b$. In other words, $\lim_s \mathtt{state}(a_0, x\upharpoonright s)$ diverges. □

One can also see that $\lim_{s\to\infty} \mathtt{state}(a, x\upharpoonright s)$ converges for every a iff the a-th strategy acts at most finitely often for any a. We finally show the following.

Lemma 15. *For any x, we have the following.*

$$x \in U \iff \psi(x) \in f^{-1}[V].$$

Proof. ($\Leftarrow$) If $x \notin U$, then by Lemma 14, $\lim_s \mathtt{state}(a, x\upharpoonright s)$ converges for every a. Then, for any a, there is a stage $\alpha_a \sqsubseteq x$ such that the a-th strategy proceeds the initial action at stage α_a and this action is never injured. Clearly, $(\alpha_a)_{a\in\omega}$ is strictly increasing. Then, by the initial action $(**)$ of a at stage α_a, for all $\tau \in V_{\alpha_a-1}$ and $i < a$ we have $\exists j \forall k \neg Q(\psi(x), \tau, i, j, k)$ since $\psi(x) \in T_{\alpha_a}$ by the non-injury assumption, and $\psi(x)$ extends $p^{\alpha_a}_{\alpha_a}$. This means that $\psi(x) \notin f_i^*[V_{\alpha_a-1}]$ for any $i < a$. However, if $\psi(x) \in f^{-1}[V]$ then there are i and β such that $\psi(x) \in f_i^*[V_\beta]$. Let a be such that $i < a$ and $\beta < \alpha_a$. Then, $\psi(x) \in f_i^*[V_{\alpha_a-1}]$, a contradiction. Therefore, we get $\psi(x) \notin f^{-1}[V]$.

($\Rightarrow$) If $x \in U$, then by Lemma 14, there is a such that the a-th strategy acts infinitely often. Let a be the least such strategy. Then there is s such that a is never injured after $x \upharpoonright s$. Assume that $\mathcal{T}_{x\upharpoonright s}$ is of the form $(T^s_u, \sigma^s_u, i^s_u)_{u<\ell(s)}$, where we must have $a < \ell(s)$. Since a and hence any $u \leq a$ are never injured after $x \upharpoonright s$, we have that $\sigma^s_u = \sigma^t_u$ and $i^s_u = i^t_u$ whenever $s \leq t$ and $u \leq a$. By the b-th action $(*)$ of a, we have $\forall j \leq b \exists k Q(\psi(x), \sigma^s_a, i^s_a, j, k)$ since $\psi(x) \in T_{\alpha_a}$ by the non-injury assumption, and $\psi(x)$ extends $p^{x\upharpoonright s}_{x\upharpoonright s}$. Since this holds for any b, we get $\forall j \exists k Q(\psi(x), \sigma^s_a, i^s_a, j, k)$. This means that $\psi(x) \in f^*_{i^s_a}[\sigma^s_a]$, and thus $\psi(x) \in f^*_{i^s_a}[V_{x\upharpoonright s}] \subseteq f^{-1}[V]$ since $\sigma^s_a \in V_{x\upharpoonright s}$. □

Lemma 15 shows that $f^{-1}[V]$ is $\mathbf{\Sigma}^0_3$-complete for some open set V. Hence, $f^{-1}[\omega^\omega \setminus V]$ is not $\mathbf{\Sigma}^0_3$ while $\omega^\omega \setminus V$ is $\mathbf{\Sigma}^0_2$. That is, $f^{-1}\mathbf{\Sigma}^0_2 \not\subseteq \mathbf{\Sigma}^0_3$. This concludes the proof. □

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