A short introduction to μ-calculus

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The modal μ-calculus

Definition

The modal μ-formulas are generated by the following grammar:

$$\begin{aligned}\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \\ \mid \Diamond\varphi \mid \Box\varphi \mid \mu X.\varphi \mid \nu X.\varphi\end{aligned}$$

P is taken from a fixed set of propositions and X is taken from a set of variables. To define negation on the μ-calculus we let it to follow the usual rules on connectives and modalities, and define

$$\neg \mu X.\varphi = \nu X.\neg\varphi[X \mapsto \neg X]$$

Models of the modal μ-calculus

The models of the modal μ-calculus are the same as the models of modal logic, i.e., labeled transition systems.

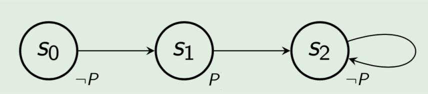
Definition

A labeled transition system is a triple $S = (S, E, \rho)$ where

- S is the set of states,
- $E \subseteq S \times S$ are the transitions, and
- $\rho : Prop \to \mathcal{P}(S)$ assigns to each proposition P the states in which P is valid.

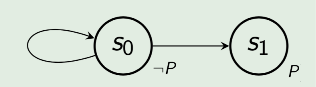
Example

$S_1 = \{s_0, s_1, s_2\}$, $E = \{s_0 \to s_1, s_1 \to s_2, s_2 \to s_2\}$ and $\rho(P) = \{s_1\}$.



Example

$S_2 = \{s_0, s_1\}$, $E = \{s_0 \to s_0, s_0 \to s_1\}$ and $\rho(P) = \{s_1\}$.



Extentional Semantics

Definition

Given a transition system S and a valuation $V : Var \to \mathcal{P}(S)$, we define

$$\|P\|_V^S = \rho(P)$$
$$\|X\|_V^S = V(X)$$
$$\|\neg\varphi\|_V^S = S \setminus \|\varphi\|_V^S$$
$$\|\varphi \wedge \psi\|_V^S = \|\varphi\|_V^S \cap \|\psi\|_V^S$$
$$\|\Box\varphi\|_V^S = \{s | \forall t \in S.\langle t, s\rangle \in E \implies s \in \|\varphi\|_V^S\}$$
$$\|\mu X.\varphi\|_V^S = \bigcup\{U \subseteq S | U \subseteq \|\varphi\|_{V[Z \to U]}^S\}$$

in the above definition, $V[Z \to U](X) = U$ if $X = Z$ and $V[Z \to U](X) = V(X)$ otherwise.

Game Semantics

Definition

Given a transition system S, a stage $s_0 \in S$, a valuation $V : Var \to \mathcal{P}(S)$ and a μ-calculus formula φ we define the game $\mathcal{G}_V^S(s_0, \varphi)$:

- The game vertices the pairs $\langle s, \psi \rangle$ where $s \in S$ and ψ is a subformula of φ.
- The initial state is $\langle s_0, \varphi \rangle$.

Game Semantics

Definition (Cont.)

- Players have the following plays:
  - $\langle s, \psi_0 \wedge \psi_1 \rangle \to \langle s, \psi_0 \rangle$, $\langle s, \psi_0 \wedge \psi_1 \rangle \to \langle s, \psi_1 \rangle$ are edges.
  - $\langle s, \psi_0 \vee \psi_1 \rangle \to \langle s, \psi_0 \rangle$, $\langle s, \psi_0 \vee \psi_1 \rangle \to \langle s, \psi_1 \rangle$ are edges.
  - If $\langle s, t \rangle \in E$, then $\langle s, \Box\psi \rangle \to \langle t, \psi \rangle$ is an edge.
  - If $\langle s, t \rangle \in E$, then $\langle s, \Diamond\psi \rangle \to \langle t, \psi \rangle$ is an edge.
  - If $\mu X.\psi$ is a subformula of φ then $\langle s, \mu X.\psi \rangle \to \langle s, \psi \rangle$ and $\langle s, X \rangle \to \langle s, \psi \rangle$ are edges.
  - If $\nu X.\psi$ is a subformula of φ then $\langle s, \nu X.\psi \rangle \to \langle s, \psi \rangle$ and $\langle s, X \rangle \to \langle s, \psi \rangle$ are edges.
- V owns $\langle s, \psi_0 \vee \psi_1 \rangle$, $\langle s, \Diamond\psi \rangle$, $\langle s, P \rangle$ if $s \notin \rho(P)$ and $\langle s, Z \rangle$ if $s \notin V(Z)$.
- R owns $\langle s, \psi_0 \wedge \psi_1 \rangle$, $\langle s, \Box\psi \rangle$, $\langle s, P \rangle$ if $s \in \rho(P)$ and $\langle s, Z \rangle$ if $s \in V(Z)$.
- The ownership of the other vertices doesn't matter.

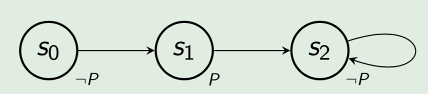
Definition (Cont.)

- If a player can't make a move, he loses.
- In an infinite play, if the outernmost infinitely many times repeated operator is μ, V loses.
- In an infinite play, if the outernmost infinitely many times repeated operator is ν, V wins.
- If V has a winning strategy we state $s_0 \models_V^S \varphi$.

Game Semantics

Example

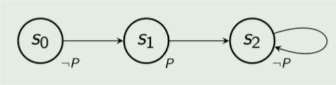
Let $\varphi = \mu X.P \vee \Diamond X$. Intuitively, this means "eventually P holds". Let

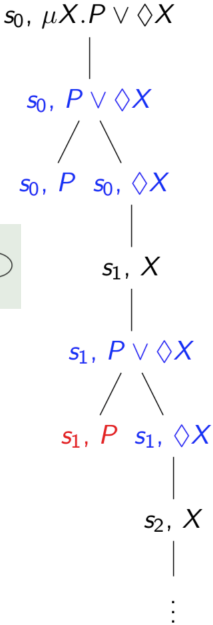


Here, $s_0 \models \varphi$.

Verifier

Refuter

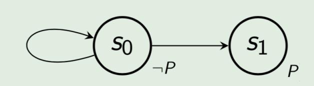




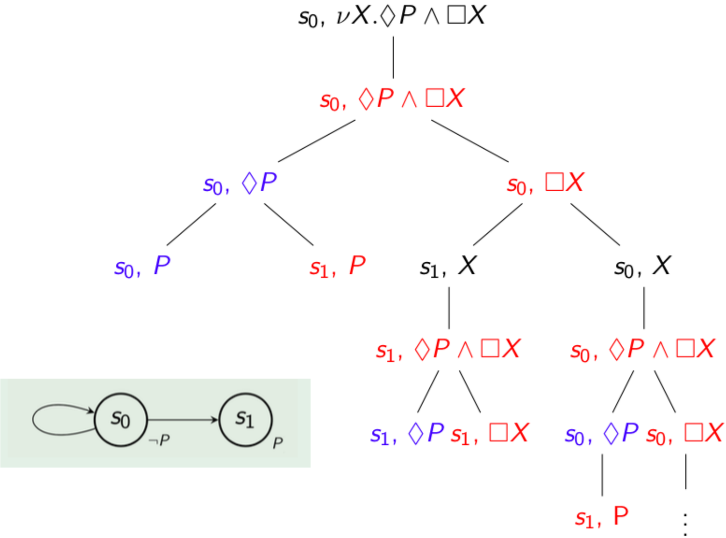
Game Semantics

Example

Let $\varphi = \nu X.\Diamond P \wedge \Box X$. Intuitively, this means "$\Diamond P$ always holds". Let



Here, $s_0 \not\models \varphi$.


$s_0,\ \nu X.\Diamond P \wedge \Box X$
$s_0,\ \Diamond P \wedge \Box X$
$s_0,\ \Diamond P$
$s_0,\ \Box X$
$s_0,\ P$
$s_1,\ P$
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$s_0,\ X$
$s_1,\ \Diamond P \wedge \Box X$
$s_0,\ \Diamond P \wedge \Box X$
$s_1,\ \Diamond P$
$s_1,\ \Box X$
$s_0,\ \Diamond P$
$s_0,\ \Box X$
$s_1,\ P$
$\vdots$
s_0
s_1
$\neg P$
P

Semantics

Theorem

The extentional semantics and game semantics are equivalent.

The Alternation Hierarchy

Definition

The simple alternation hierarchy is defined by:

- Σ_0^μ, Π_0^μ: the class of formulas with no fixpoint operators.
- Σ_{n+1}^μ : the class of formulas containing $\Sigma_n^\mu \cup \Pi_n^\mu$ and closed under the operations $\vee, \wedge, \Box, \Diamond$ and μx.
- Π_{n+1}^μ : the class of formulas containing $\Sigma_n^\mu \cup \Pi_n^\mu$ and closed under the operations $\vee, \wedge, \Box, \Diamond$ and νx.
- $\Delta_n^\mu := \Sigma_n^\mu \bigcap \Pi_n^\mu$

The Alternation Hierarchy

Example

- $\mu X.P \vee \Diamond X$ is Σ_1^μ.
- $\nu X.\Diamond P \wedge \Box X$ is Π_1^μ.
- $\mu X.(\nu X.\Diamond P \wedge \Box X) \vee \Diamond X$ is Σ_2^μ
- $\mu X_1.\nu X_2.\mu X_3.(X_1 \wedge X_2 \wedge X_3)$ is Σ_3^μ.

The Alternation Hierarchy

Definition

The semantic hierarchy is defined as:

$$\Sigma_n^\mu = \{ \|\varphi\|^T | \varphi \in \Sigma_n^\mu \wedge \ T \text{ is a transition system} \}$$

Theorem (Bradfield)

The semantic hierarchy theorem is a proper hierarchy, i.e.,

$$\Sigma_n^\mu \subsetneq \Sigma_{n+1}^\mu \text{ for all } n$$

Theorem (Bradfield)

The semantic hierarchy theorem for finite transition systems is a proper hierarchy.

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The μ-arithmetic

Definition

- We can define the μ-arithmetic by adding set variables and the fixpoint operator μ to PA.
- We can then form the set term $\mu x X.\varphi$. The μ operator binds x and X. (We have a restriction on φ to be X-positive, but we omit this definition here.)
- $\mu x X.\varphi$ is the defined to be the least fixed point of the operator

$$\Gamma_\varphi(X) = \{x|\varphi(x, X)\}.$$

The arithmetic μ-calculus

Example

The following formula defines the even numbers in the μ-calculus:

$$\mu x X.(x = 0 \vee (x-2) \in X)$$

Calculating the least fixed point of $\Gamma_{x=0\vee(x-2)\in X}$ we have

$$\emptyset \mapsto \{0\} \mapsto \{0,2\} \mapsto \{0,2,4\} \mapsto \cdots \mapsto \{0,2,4,6,8,\cdots\}$$

The Alternation Hierarchy (Arithmetic Version)

Definition

- Σ_0^μ, Π_0^μ: the class of first order formulas and set variables
- Σ_{n+1}^μ : the class of formulas containing $\Sigma_n^\mu \cup \Pi_n^{\mathrm{W}\mu}$ and closed under the first order connectives and forming $\mu X.\varphi$ for $\varphi \in \Sigma_n^\mu$.
- Π_{n+1}^μ : the class of formulas containing $\Sigma_n^\mu \cup \Pi_n^{\mathrm{W}\mu}$ and closed under the first order connectives and forming $\nu X.\varphi$ for $\varphi \in \Pi_n^\mu$.
- $\Delta_n^\mu := \Sigma_n^\mu \bigcap \Pi_n^\mu$

The Alternation Hierarchy (Arithmetic Version)

Theorem (Lubarsky)

Any Σ_n^μ-formula can be put in the form

$$\tau_n \in \mu X_n.\tau_{n-1} \in \nu X_{n-1}.\tau_{n-2} \in \mu X_{n-2}.\dots.\tau_1 \in \eta X_1.\varphi$$

where φ is a first order formula.

Theorem (Bradfield)

The alternation hierarchy for the μ-arithmetic is strict.

Proof Idea

We define for each Σ_n^μ a satisfaction formula. We do this similarly to defining the partial satisfaction formulas of PA. (Remember to make use of the normal forms.)

The Alternation Hierarchy (Arithmetic Version)

Theorem (Bradfield)

For each modal μ-calculus formula $\varphi \in \Sigma_n^\mu$ and for each recursively presentable transition system T, $\|\varphi\|^T$ is Σ_n^μ-definable set of integers.

Theorem (Bradfield)

Let $\varphi(z)$ be a Σ_n^μ formula of μ-arithmetic. There is a r.p.t.s. T, a valuation V and a Σ_n^μ modal μ-formula $\overline{\varphi}$ such that $\varphi((s)_0)$ iff $s \in \|\overline{\varphi}\|_S^T$. Thus if φ is not Σ_n^μ-definable, neither is $\|\overline{\varphi}\|$.

We can combine these theorems with the strictness of the alternation hierarchy for the arithmetic μ-calculus to get a proof of the alternation hierarchy for the modal μ-calculus. (Indeed, this is the original proof by Bradfield.)

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Definition

Transfinite μ-arithmetic We can extend the definition of the alternation hierarchy for μ-arithmetic by defining Σ^{μ}_{λ} to be the set of formulas

$$\bigvee_{i<\omega} \varphi_i$$

where the φ_i can be recursively enumerated and each φ_i is in some $\Sigma^{\mu}_{\beta_i}$ with $\beta_i < \lambda$. We do this for $\lambda < \omega_1^{ck}$.

μ-calculus and the difference hierarchy

Let Σ^δ_α be the α-th level of the difference hierarchy of Σ^0_2.
$\Game$ is the game quantifier:
$\Game\alpha.P(\alpha,\vec{x}) = \{\vec{x} |\ I$ wins the Gale-Steward game with payoff $P(\alpha,\vec{x})\}$

Theorem (Bradfield, Duparc, Quickert)

For all $\alpha < w_1^{ck}$, $\Game\Sigma^\delta_\alpha = \Sigma^\mu_{\alpha+1}$.

Furthermore as

Theorem (MedSalem, Tanaka)

$\bigcup_{\alpha<\omega_1^{ck}} \Sigma^\delta_\alpha = \Delta^0_3$

we can show

Corollary

$\Sigma^\mu_{\omega_1^{ck}} = \Game\Delta^0_3.$

For context, we have:

Theorem (Kechris-Moschovakis)

$\Sigma_0^\mu = \Game\Sigma_1^0 = \Pi_1^1$

Theorem (Solovay)

$\Sigma_1^\mu = \Game\Sigma_2^0 = \Sigma_1^1\text{-}IND$

A result on reverse math

Definition

We can define the μ-arithmetic as a subsystem of second arithmetic by considering as axioms:

- ACA_0 and
- $\mu x X.\varphi(x, X)$ is the least fixed point of Γ_φ for all adequate φ. Note: we are going to skip the formalization here.

Definition

$< \omega$-$\mathbf{\Sigma}^0_2$-Det is the subsystem of second order arithmetic that says that all (finite) Boolean combinations of $\mathbf{\Sigma}^0_2$ sets are determined.

A result on reverse math

Theorem (Heinatsch, Möllerfeld)

Over ACA_0, μ-arithmetic and $< \omega$-$\mathbf{\Sigma^0_2}$-Det are equivalent over $\mathcal{L}_2$-formulas.

Theorem (Möllerfeld)

μ-arithmetic and Π^1_2-CA_0 are Π^1_1-conservative over $\mathcal{L}_2$-formulas. Therefore they are proof-theoretic equivalent.

Corollary (Heinatsch, Möllerfeld)

$< \omega$-$\mathbf{\Sigma^0_2}$-Det and Π^1_2-CA_0 are Π^1_1-conservative over $\mathcal{L}_2$-formulas. Therefore they are proof-theoretic equivalent.

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