Quantum Physics and Logic 2022, Oxford, 27 June – 1 July 2022

Logical Characterization of Contextual Hidden-Variable Theories Based on Quantum Set Theory

Masanao Ozawa

Chubu University and Nagoya University

Introduction

- **To make quantum logic great, we construct mathematics based on quantum logic.**
- **According to Bourbaki, we have only to construct set theory instead of constructing various branches of mathematics.**
- **For this purpose, Gaisi Takeuti introduced quantum set theory (QST) in 1981.**
- **Takeuti showed that observables (self-adjoint operators) A of the quantum system bijectively correspond to the internal real numbers $\tilde{A}$ in QST. We generalized Takeuti's QST for arbitrary von Neumann algebras and use them to characterize contextual hidden-variable theories.**

Quantum set theory

- **Denote by $L(\in)$ the language of set theory without constants, and by $L(\in, U)$ the language of set theory augmented by constants denoting elements of U.**

- **To any von Neumann algebra $\mathcal{M}$ with the projection lattice $\mathcal{P}(\mathcal{M})$, we constructed a set theoretical universe $V[\mathcal{M}]$ such that:**

  (i) For any statement $\phi(u_1, \ldots, u_n) \in L(\in, V[\mathcal{M}])$, the $\mathcal{P}(\mathcal{M})$-valued truth value $[\![\phi(u_1, \ldots, u_n)]\!]$ is assigned.

(ii) Δ_0-Elementary Quantum Extension: $V[\mathcal{M}]$ extends the standard universe V of ZFC set theory by embedding $a \in V \mapsto \check{a} \in V[\mathcal{M}]$ such as

(a) $a \in b$ if and only if $[\![\check{a} \in \check{b}]\!] = 1$,

(b) $a = b$ if and only if $[\![\check{a} = \check{b}]\!] = 1$,

(c) $\phi(a_1, \ldots, a_n)$ holds in V if and only if $[\![\phi(\check{a}_1, \ldots, \check{a}_n)]\!] = 1$.

(iii) Takeuti Correspondence: The self-adjoint operators A affiliated with $\mathcal{M}$ bijectively correspond to the internal real numbers $\tilde{A}$ in $V[\mathcal{M}]$ satisfying

$$[\![\tilde{A} \leq \check{\lambda}]\!] = E^A(\lambda)$$

(iv) Transfer Principle: For any Δ_0-formula $\phi(x_1, \ldots, x_n)$ provable in ZFC set theory and for any $u_1, \ldots, u_n \in V[\mathcal{M}]$,

$$[\![\phi(u_1, \ldots, u_n)]\!] \geq \underline{\vee}(u_1, \ldots, u_n),$$

where $\underline{\vee}(u_1, \ldots, u_n)$ is the $\mathcal{P}(\mathcal{M})$-valued commutators of $u_1, \ldots, u_n \in V[\mathcal{M}]$.

(v) $\mathcal{M}$ is abelian if and only if

$$[\![\phi(u_1, \ldots, u_n)]\!] = 1$$

for any formula $\phi(x_1, \ldots, x_n) \in L(\in)$ provable in ZFC set theory and any $u_1, \ldots, u_n \in V[\mathcal{M}]$.

Algebraic characterization

- **Bohr's notion of elements of physical reality suggested in the Bohr-EPR debate:**

  (i) (Classicality) They should be described by a classical language obeying classical logic.

  (ii) (A-Priv) If A is measured, A is an element of reality.

  (iii) (Contextualized EPR criterion) If the value of B is inferred with probability unity by the result of A-measurement without disturbing the value of B (i.e., $[A, B] = 0$), B is also an element of reality.

- **Definition. Given a measurement context (ψ, A), consisting of a state vector ψ and an observable A, an observable considered to have its value is called a beable in (ψ, A).**

- **Halvorson and Clifton maximally characterized the beables in (ψ, A) as observables affiliated with the following von Neumann algebra $\mathcal{M}$, called a maximal beable subalgebra of $\mathcal{L}(\mathcal{H})$ definable by (ψ, A).**

  (i) (Beable) $\mathcal{M}$ is beable in ψ, i.e., there exists a probability measure μ on the space $\mathcal{D}(\mathcal{M})$ of dispersion-free states of $\mathcal{M}$ such that

$$(\psi, X\psi) = \int_{\mathcal{D}(\mathcal{M})} \omega(X) d\mu(\omega),$$

  for any $X \in \mathcal{M}$.

  (ii) (A-Priv) A is affiliated with $\mathcal{M}$.

  (iii) (Definability) $\mathcal{M}$ is implicitly definable by (ψ, A), i.e., $U^*\mathcal{M}U = \mathcal{M}$ for every unitary operator U on $\mathcal{H}$ such that $U\psi = \psi$ and $U^*AU = A$.

  (iv) (Maximality) $\mathcal{M}$ is maximal among all von Neumann algebras $\mathcal{M}$ satisfying (i)–(iii).

- **Theorem (Halvorson-Clifton 1999): For any measurement context (ψ, A) there uniquely exists a maximal beable subalgebra definable in (ψ, A).**

Logical characterization

- **The language $L(\in, V[\mathcal{M}])$ is called ZFC-satisfiable in a state ψ iff any formula $\phi(x_1, \ldots, x_n) \in L(\in)$ provable in ZFC holds with probability unity for any $u_1, \ldots, u_n \in V[\mathcal{M}]$, i.e.,**

$$(\psi, [\![\phi(u_1, \ldots, u_n)]\!]\psi) = 1.$$

- **The language $L(\in, V[\mathcal{M}])$ is called a maximal ZFC-satisfiable theory definable by a measurement context (ψ, A) iff it satisfies**

  (i) (ZFC-Satisfiability) $L(\in, V[\mathcal{M}])$ is ZFC-satisfiable in ψ.

  (ii) (A-Priv) $\tilde{A}$ is an internal real number in $L(\in, \mathcal{M})$.

(iii) (Definability) $L(\in, V[\mathcal{M}])$ **is implicitly definable by** (ψ, A)**, i.e.,** $\alpha_U(V[\mathcal{M}]) = V[\mathcal{M}]$ **for any unitary** U **such that** $U\psi = \psi$ **and** $U^*AU = A$**, where** $\alpha_U(u)$ **is defined for all** $u \in V[\mathcal{M}]$ **by the following recursion:**

$$\alpha_U(u) = \{(\alpha_U(x), U^*u(x)U) \mid x \in \mathrm{dom}(u)\}.$$

(iv) (Maximality) $L(\in, V[\mathcal{M}])$ **is maximal among all languages** $L(\in, V[\mathcal{M}])$ **satisfying (i)–(iii).**

- **Theorem: The language $L(\in, V[\mathcal{M}])$ is ZFC-satisfiable in ψ if and only if $\mathcal{M}$ is beable in ψ.**

- **Theorem: For any measurement context (ψ, A) there uniquely exists a maximal ZFC-satisfiable theory $L(\in, V[\mathcal{M}])$ definable by a measurement context (ψ, A). In this case, $\mathcal{M}$ is the maximal beable subalgebra definable by (ψ, A)**

- **Theorem: If $\tilde{A} = \tilde{B}$ holds with probability unity in the state ψ, i.e.,**

$$(\psi, [\![\tilde{A} = \tilde{B}]\!]\psi) = 1,$$

then $\tilde{B}$ is in the maximal ZFC-satisfiable theory definable by the measurement context (ψ, A).

- **Example. Consider the composite system S of two spin 1/2 particles with the state space $\mathcal{H} = \mathrm{C}^2 \otimes \mathrm{C}^2$, in the singlet state $\psi = 2^{-1/2}(|+_z\rangle|-_z\rangle - |-_z\rangle|+_z\rangle)$ and the obsevable $A = \sigma_z \otimes 1$. Let $\mathcal{B}(\psi, A)$ be the maximal beable subalgebra of $\mathcal{L}(\mathcal{H})$ definable by (ψ, A). Let $B = 1 \otimes \sigma_z$. Then,**

$$(\psi, [\![\tilde{A} = \tilde{B}]\!]\psi) = 1,$$

so that $1 \otimes \sigma_z \in \mathcal{B}(\psi, A)$ but $1 \otimes (\cos\theta\sigma_z + \sin\theta\sigma_x) \notin \mathcal{B}(\psi, A)$ if $\cos\theta \neq \pm 1$.

- **Conclusion: We conclude that for any measurement context (ψ, A) there uniquely exists a maximal contextual hidden-variable theory, i.e., the maximal ZFC-satisfiable theory definable by the measurement context (ψ, A), which maximally realizes Bohr's notion of elements of physical reality.**